Classical Yang-Baxter equation from supergravity

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Araujo, IB, Ó Colgáin, Kelekçi, Sakamoto, Sheikh-Jabbari, Yavartanoo, Yoshida
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Statement

For any supergravity solution there exists a deformation, where the field equations reduce to the classical Yang-Baxter equation.

- Supergravity: $\mathcal{N} = 2$ $d = 10$ (type II);
- Solution: metric $G_{\mu\nu}$, dilaton Φ, RR fields. No $B_{\mu\nu}$;
- Deformation: is based on the isometry group of $G_{\mu\nu}$.
- Deformation parameter: r-matrix, which turns out to be constrained by the classical YB equation.

String integrability and Yang-Baxter deformation

- Green-Schwarz superstring σ-model on $AdS_5 \times S^5$ is classically integrable [Bena, Polchinski, Roiban 2003];
- The crucial evidence for AdS/CFT duality between superstring theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super-Yang-Mills on $\mathbb{R}^{1,3}$.

Integrability preserving deformations:

- T-duality-shift-T-duality (TsT) [Lunin, Maldacena; Frolov];
- Yang-Baxter deformations [Klimcik; Delduc, Magro, Vicedo; Kawaguchi, Matsumoto, Yoshida].

The Yang-Baxter deformations are parameterised by an **r-matrix**, which solves the classical homogeneous Yang-Baxter equation.

Yang-Baxter deformations

Coset formulation [Metsaev-Tseytlin] of the Yang-Baxter deformed AdS_5 σ-model:

$$\mathcal{L} = \mathrm{Tr}\left[A\,P^{(2)} \circ \frac{1}{1 - 2\eta R_g \circ P^{(2)}} A\right], \quad A = -g^{-1}dg, \quad g \in SO(4,2),$$

where

$$R_g(X) = g^{-1}R(gXg^{-1})g, \quad X \in \mathfrak{so}(4,2),$$

R is an antisymmetric operator satisfying the homogeneous classical Yang-Baxter equation,

$$[R(X), R(Y)] - R([R(X), Y] + [X, R(Y)]) = 0, \quad X, Y \in \mathfrak{so}(4,2).$$

r-matrix parameterisation of the YB deformation

Parameterise R in terms of the **r-matrix**:

$$R(X) = \mathrm{Tr}_2[r(1 \otimes X)] = \sum_{i,j} r^{ij} b_i \mathrm{Tr}[b_j X],$$

$$r = \frac{1}{2} \sum_{i,j} r^{ij} b_i \wedge b_j, \quad b_i \in \mathrm{bas}\, \mathfrak{so}(4,2).$$

In terms of the small r-matrix the classical Yang-Baxter equation takes the form:

$$f_{ij}{}^{a} r^{ib} r^{jc} + f_{ij}{}^{c} r^{ia} r^{jb} + f_{ij}{}^{b} r^{ic} r^{ja} = 0,$$

where f_{ij}^k are structure constants,

$$[b_i, b_j] = f_{ij}^k b_k.$$

Two views of Yang-Baxter deformations

"Standard" narration:

- a string σ-model in **coset formalism** can be deformed by introducing an r-matrix-dependent operator into the σ-model;
- **r-matrix** solving the YB equation is viewed as an input;
- deformed spacetime may be a solution to generalised supergravity;
- YB deformed supergravity solutions can be obtained via TsT transformations or non-abelian T-duality.

Our narration:

- Deformation is the Seiberg-Witten **open-closed string map**, essentially matrix inversion;
- procedure works for non-coset geometries, e.g. Schwarzschild or Bianchi cosmology;
- field equations for the deformed background **imply** the YB equation.

Our approach

For a type II supergravity solution with the metric $G_{\mu\nu}$, the Killing vectors K_i^μ, and the isometry algebra $[K_i, K_j] = f_{ij}{}^k K_k$:

- Choose the r-matrix coefficients r^{ij} and define $\Theta^{\mu\nu} = r^{ij} K_i^\mu K_j^\nu$;
- Open-closed string map: $g_{\mu\nu} + b_{\mu\nu} = \left(\frac{1}{G^{-1}+\Theta}\right)_{\mu\nu}$;
- Prescription for the deformed dilaton and the RR fields.

Field equations for $g_{\mu\nu}$, $b_{\mu\nu}$, etc. then **imply** that the r-matrix satisfies the Yang-Baxter equation:

$$f_{ij}{}^a r^{ib} r^{jc} + f_{ij}{}^c r^{ia} r^{jb} + f_{ij}{}^b r^{ic} r^{ja} = 0.$$

$AdS_2 \times S^2 \times T^6$ example

The original background:

$$ds^2 = \frac{-dt^2 + dr^2}{r^2} + d\theta^2 + \sin^2\theta d\phi^2.$$

A simple Ansatz for the deformation parameter:

$$\Theta^{tr} = \Theta_1(t, r), \qquad \Theta^{\theta\phi} = \Theta_2(\theta, \phi).$$

Find the deformed metric $g_{\mu\nu} = (G^{-1} + \Theta)^{-1}{}_{(\mu\nu)}$, solve the Einstein equations:

$$\Theta_1 = c_1 tr + c_2 r(t^2 - r^2) + c_3 r + c_4 r^2,$$
$$\Theta_2 = c_5 \cos\phi + c_6 \sin\phi + c_7 \cot\theta + \frac{c_8}{\sin\theta}.$$

CYBE emerges as the constraints on the r-matrix coefficients

$$c_1^2 - 4c_2c_3 = 0 = c_5^2 + c_6^2 + c_7^2, \qquad c_4 = 0 = c_8.$$

Lessons from the $AdS_2 \times S^2$ example

- We have not assumed that $\Theta \sim K_i \wedge K_j$ from the beginning, rather this is enforced by the field equations as follows:
- Denote AdS_2 Killing vectors,

$$K_1 = -t\partial_t - r\partial_r, \quad K_2 = -\partial_t, \quad K_3 = -(t^2 + r^2)\partial_t - 2tr\partial_r.$$

- Then $\Theta_1 = (c_1 tr + c_2 r(t^2 - r^2) + c_3 r)\partial_t \wedge \partial_r$ is equal to the Killing bivector r-matrix:

$$r = -c_3 K_1 \wedge K_2 + \frac{c_1}{2} K_2 \wedge K_3 - c_2 K_3 \wedge K_1,$$

- the constraint on c_1, c_2, c_3 is then nothing but the classical YB equation $\mathfrak{sl}(2)$:

$$f_{ij}{}^a r^{ib} r^{jc} + f_{ij}{}^c r^{ia} r^{jb} + f_{ij}{}^b r^{ic} r^{ja} = 0.$$

Bianchi type III spacetime

$$ds^2 = -(a_1 a_2 a_3\, e^{-2\Phi} dt)^2 + (a_1 dx)^2 + (a_2 dy)^2 + (a_3 e^x dz)^2, \qquad \Phi = \lambda t,$$

$$a_1 = a_3 = \frac{p_1}{\sinh(p_1 t)} e^{-\frac{1}{2} p_2 t + \lambda t}, \quad a_2 = e^{\frac{1}{2} p_2 t + \lambda t}, \qquad 4p_1^2 = p_2^2 + 4\lambda^2.$$

Isometry algebra:

$$K_1 = \partial_x - z\partial_z, \quad K_2 = \partial_y, \quad K_3 = \partial_z, \qquad [K_1, K_3] = K_3.$$

The most general r-matrix

$$r = \alpha K_1 \wedge K_2 + \beta K_2 \wedge K_3 + \gamma K_3 \wedge K_1$$

is a solution to the CYBE provided $\alpha\gamma = 0$.

Bianchi III spacetime

When $\gamma = 0$, it can be checked that the deformed geometry

$$g_{\mu\nu}dx^\mu dx^\nu = -(a_1 a_2 a_3 e^{-2\lambda t} dt)^2 + \frac{1}{[1+\alpha^2 a_2^2(a_1^2 + z^2 e^{2x} a_3^2)]}\Big[a_1^2 dx^2 + a_2^2 dy^2 + a_3^2 e^{2x} dz^2 + \alpha^2 e^{2x} a_1^2 a_2^2 a_3^2 (zdx + dz)^2\Big],$$

$$b = -\frac{\alpha a_2^2}{[1+\alpha^2 a_2^2(a_1^2 + z^2 e^{2x} a_3^2)]}(a_1^2 dx \wedge dy + z e^{2x} a_3^2 dy \wedge dz),$$

$$\Phi = \lambda t - \frac{1}{2}\log[1+\alpha^2 a_2^2(a_1^2 + z^2 e^{2x} a_3^2)],$$

is a solution to supergravity.

Comments

- The proposed map is defined for any spacetimes (not only for cosets);
- Classical Yang-Baxter equation is **an output** rather than the input.
- Solving field equations for $(g_{\mu\nu}, b_{\mu\nu})$ for an arbitrary Θ is intractable; for practical purposes we assumed that $\Theta = r^{ij} K_i K_j$.

The above examples seemed compelling to us, but they do not prove that CYBE always emerges.

Perturbative proof

Assume that the NC parameter is an arbitrary Killing bivector,

$$\Theta^{\mu\nu} = r^{ij} K_i^\mu K_j^\nu, \qquad r^{ij} = -r^{ji}.$$

Expand everything in powers of Θ:

$$\begin{aligned} g_{\mu\nu} &= G_{\mu\nu} + \Theta_\mu{}^\alpha \Theta_{\alpha\nu} + \mathcal{O}(\Theta^4), \\ B_{\mu\nu} &= -\Theta_{\mu\nu} - \Theta_{\mu\alpha}\Theta^{\alpha\beta}\Theta_{\beta\nu} + \mathcal{O}(\Theta^5), \\ \phi &= \Phi + \frac{1}{4}\Theta_{\rho\sigma}\Theta^{\rho\sigma} + \mathcal{O}(\Theta^4). \end{aligned}$$

Write down the field equations.
Second order in Θ: dilaton field equation

$$K_i^\alpha K_k^\beta \nabla_\alpha K_{\beta m} \left(f_{l_1 l_2}{}^m r^{il_1} r^{kl_2} + f_{l_1 l_2}{}^k r^{ml_1} r^{il_2} + f_{l_1 l_2}{}^i r^{kl_1} r^{ml_2} \right) + \left(\Theta^{\beta\gamma}\Theta^{\alpha\lambda} + \Theta^{\alpha\beta}\Theta^{\gamma\lambda} + \Theta^{\gamma\alpha}\Theta^{\beta\lambda} \right) R_{\beta\gamma\alpha\lambda} = 0.$$

Summary

- Yang-Baxter deformed supergravity solutions can be derived in a much simpler way than in the traditional approach.
- Dynamics of supergravity fields reduces to an algebraic equation.
- A potentially powerful (generalised) supergravity solution generating technique.
- The deformation prescription is valid for non-coset and for non-integrable backgrounds.
- Classification of solutions to CYBE from gravity?
- What is the role of integrability?

Yang-Baxter deformations

Coset formulation of the Yang-Baxter deformed $AdS_5 \times S^5$ σ-model:

$$\mathcal{L} = \mathrm{Tr}\left[A\,P^{(2)} \circ \frac{1}{1 - 2\eta R_g \circ P^{(2)}} A\right], \quad A = -g^{-1}dg, \quad g \in SO(4,2),$$

where

$$P^{(2)}(X) = \eta^{mn}\mathrm{Tr}[X\,\mathbf{P}_m]\,\mathbf{P}_n, \quad X \in \mathfrak{so}(4,2), \quad \mathbf{P}_m \in \mathrm{bas}\,\frac{\mathfrak{so}(4,2)}{\mathfrak{so}(4,1)},$$
$$R_g(X) = g^{-1}R(gXg^{-1})g,$$

R is an antisymmetric operator satisfying the homogeneous classical Yang-Baxter equation,

$$[R(X), R(Y)] - R([R(X), Y] + [X, R(Y)]) = 0, \quad X, Y \in \mathfrak{so}(4,2).$$

Examples of YB deformed backgrounds: $r = \frac{1}{2} P_1 \wedge P_2$

Starting from $AdS_5 \times S^5$:

$$ds^2_{\rm open} = \frac{1}{z^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dz^2) + ds^2(S^5),$$

using the abelian r-matrix $r = \frac{1}{2} P_1 \wedge P_2$:

$$ds^2 = \frac{-dt^2 + dx_3^2 + dz^2}{z^2} + \frac{z^2}{z^4 + \eta^2}(dx_1^2 + dx_2^2) + ds^2(S^5),$$

$$B = \frac{\eta}{z^4 + \eta^2} dx_1 \wedge dx_2, \quad e^{2\Phi} = g_s^2 \frac{z^4}{z^4 + \eta^2}.$$

[Hashimoto, Itzhaki; Maldacena, Russo]

Generalised IIB supergravity

[Arutyunov, Frolov, Hoare, Roiban, Tseytlin]

For less trivial r-matrices, deformed background is not a solution of supergravity.

$$R_{\mu\nu} = \frac{1}{4} H_{\mu\rho\sigma} H_{\nu}{}^{\rho\sigma} - \nabla_\mu X_\nu - \nabla_\nu X_\mu,$$

$$R = \frac{1}{12} H^2 - 4\nabla_\mu X^\mu + 4 X_\mu X^\mu,$$

$$\frac{1}{2}\nabla^\rho H_{\rho\mu\nu} = X^\rho H_{\rho\mu\nu} + \nabla_\mu X_\nu - \nabla_\nu X_\mu.$$

There are also deformed field equations for the RR sector.

I^μ is a Killing vector

$$X_\mu = \partial_\mu \Phi + I^\nu (G_{\nu\mu} + B_{\nu\mu})$$

Divergence condition

Some YB deformations result in solutions of generalised supergravity, which are specified by a Killing vector field I^μ [Arutyunov, Frolov, Hoare, Roiban, Tseytlin]

The recipe for construction of the vector field I:

$$I^\mu = \nabla_\nu \Theta^{\nu\mu}.$$

- Directly relates open and closed string pictures;
- The condition has been attributed to the preservation of the Λ-symmetry in the generalised supergravity;
- Initially observed for examples, but later has been proven perturbatively (i.e. also follows from the field equations).

Can also be viewed as a bulk-boundary relationship for the NC parameter (supporting the holographic NC idea).

Schwarzschild

The previous example was a simple coset model.

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2(d\zeta^2 + \sin^2\zeta\, d\chi^2).$$

Fix Θ to be a Killing bivector (this involves ∂_t and the three vectors on the sphere):

$$\begin{aligned}
\Theta^{t\zeta} &= -\epsilon\cos\chi + \lambda\sin\chi, \\
\Theta^{t\chi} &= \delta + \cot\zeta\,(\epsilon\sin\chi + \lambda\cos\chi), \\
\Theta^{\zeta\chi} &= \alpha\cos\chi - \beta\cot\zeta + \gamma\sin\chi.
\end{aligned}$$

The field equations precisely match the CYBE.