Vacuum polarization of scalar and spinor field on the homogeneous spaces

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Introduction

- Quantum field theory in a curved space–time is a sufficiently well developed theory, which attracts interest in view of relevant applications to cosmology and astrophysics. The most important quantity characterizing matter is the energy–momentum tensor (EMT). It plays the role of a source for the gravitational field in the Einstein equations and describes the coupling of matter to the gravitational field. Expectation values of the EMT in the vacuum state characterize the effect of vacuum polarization and, if the vacuum state is not defined uniquely, also the effects of particle creation by the gravitational field.
- We note that practically all the currently known models of Riemannian manifolds of general relativity are associated with various transformation groups and, not infrequently, belong to the class of homogeneous Riemannian spaces. In modern cosmology, homogeneous spaces underlie the construction of Big Bang models, initial singularities, and inflationary models. The problem of taking quantum vacuum effects on homogeneous space into account then arises naturally.

- This problem is closely related to the problem of exactly integrating relativistic wave equations on manifolds with curvature and a nontrivial topology. The most universal solution method is the method of separation of variables [*V. N. Shapovalov, Sov. Phys. J. 21 (1978), E. G. Kalnins (1986), V. G. Bagrov, D. M. Gitman (1990)*]. But it accounts for only a commutative algebra of the symmetry equations, while there exists a class of spaces that do not admit separation of variables. Therefore, in most cases, quantum vacuum effects can be calculated by imposing various constraints on the metric of the space that would allow integrating the wave field equations.
- To solve this problem, we use the method of orbits, which allows performing a noncommutative reduction (the method of noncommutative integration [*A. V. Shapovalov and I. V. Shirokov, Theor. Math. Phys., 104, 921–934 (1995)*]) of the Klein–Gordon and Dirac equations to an equation with fewer independent variables on a manifold with simpler geometry and topology. This method, unlike the method of separation of variables, takes the noncommutative symmetry algebra of the equation into account. Moreover, the solution is constructed globally and is independent of the choice of local coordinates on the homogeneous space.

Homogeneous space

Let G be a simply connected real Lie group with the Lie algebra $\mathfrak{g}$, M be a homogeneous space with right action of the group G. For any $x \in M$ there exists an isotropic subgroup $H_x \in G$. Denote by H a stationary subgroup of a point $x_0 \in M$, and let $\mathfrak{h}$ be the Lie algebra of H.

The homogeneous space M is diffeomorphic to the factor manifold G/H of right cosets of Lie group G by the isotopy subgroup H

$$M \simeq G/H. \tag{1}$$

Coordinates of an arbitrary point of $g \in G$ can be written as

$$g = hs(x), \quad h \in H, \quad g^A = (x^a, h^\alpha), \quad x = \pi(g), \tag{2}$$
$$a = 1, \dots, \dim M, \quad \alpha = \dim M + 1, \dots, \dim M,$$

where $s = \pi^{-1} : M \to G$ is a local and smooth section of G.

Some homogeneous spaces do not admitting separation of variables

1. Bianchi IX and VII static metrics:

$$ds^2 = -dt^2 + G_{ab}\sigma^a(x)\sigma^b(x), \quad G_{ab} = \text{constant}\,. \tag{3}$$

2. McLenaghan, Tariq and Tupper found a solution of the Einstein Maxwell equations with metric:

$$ds^2 = (dx^0 - 2x^2 dx^3)^2 - \frac{a^2}{(x^1)^2}\left((dx^1)^2 + (dx^2)^2\right) - (x^1)^2 (dx^3)^2, \quad a = \text{const}\,. \tag{4}$$

3. Four metrics are given in *A.Z. Petrov, Einstein Spaces, Oxford (1969) [Russian original published by Nauka, Moscow (1951)]* [Eqs.(33.45)–(33.46) on p. 312 of the Russian original].

The metric tensor of an invariant metric

The Lie algebra $\mathfrak{g}$ is decomposed into a direct sum of subspaces $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ where $\mathfrak{m} \simeq T_{x_0} M$ is a complement to $\mathfrak{h}$.

The metric tensor in local coordinates has form

$$g_{ij}(x) = G_{ab}\sigma_i^a(x, e_H)\sigma_j^b(x, e_H), \quad G_{ab} \equiv \mathbf{G}(e_a, e_b), \tag{5}$$
$$a, b, i, j = 1, \ldots, \dim M.$$

Here $\{e_a\}$ are fixed basis vectors in the space $\mathfrak{m}$, $\sigma^b(g)$ are basis right-invariant 1-forms, $\sigma^b(g) \equiv -(R_g)^* e^b$, and vectors $\{e^b\}$ are basis in the dual space $\mathfrak{m}^*$: $\langle e_a, e^b \rangle = \delta_a^b$. In the basis of the algebra $\mathfrak{g}$ the Ad_H-invariance condition takes the form

$$G_{ab}C^a_{c\alpha} + G_{ac}C^a_{b\alpha} = 0, \tag{6}$$
$$a, b, c = 1, \ldots, \dim M, \quad \alpha = 1, \ldots, \dim \mathfrak{h},$$

where $C^D_{AB} = [e_A, e_B]^D$ are the structure constant of $\mathfrak{g}$.

Klein-Gordon and Dirac equation in homogeneous space

The Klein-Gordon equation on a homogeneous space M with the invariant metric g_M has the form

$$\left(g^{ij}(x)\nabla_i\nabla_j + \zeta R + m^2\right)\psi(x) = 0, \quad (7)$$

where ζ is the conformal factor ensuring the coupling of the gravitational field to curvature, ∇_i is the covariant derivative corresponding to the Levi-Civita connection in M, m is the mass of the field, R is scalar curvature.
Denote by V_Ψ the space of spinor fields in M. We write the Dirac equation in the space M as:

$$\left(i\gamma^i(x)[\nabla_i + \Gamma_i(x)] - m\right)\psi(x) = 0. \quad (8)$$

Here $\gamma^i(x)$ are the Dirac matrices satisfying the condition

$$\{\gamma_i(x), \gamma_j(x)\} = 2g_{ij}(x)E, \quad (9)$$

where E denotes the identity matrix. For the spinor connection $\Gamma_i(x)$ an explicit formula is used

$$\Gamma_i(x) = -1/4(\nabla_i\gamma_k(x))\gamma^k(x). \quad (10)$$

We seek a solution of (9) as a tetrad decomposition

$$\gamma^i(x) = \hat{\gamma}^a \eta^i_a(x, e_H), \quad \hat{\gamma}^a = \gamma^i(x)\sigma^a_i(x, e_H). \tag{11}$$

The constant matrices $\hat{\gamma}^a$ are tetrad components of $\gamma^i(x)$ and satisfy the algebraic system of equations

$$\{\hat{\gamma}^a, \hat{\gamma}^b\} = 2G^{ab}E_4. \tag{12}$$

From (5) for γ matrices with subscripts using (5) it follows

$$\gamma_i(x) = g_{ij}(x)\gamma^i(x) = \hat{\gamma}_a \sigma^a_i(x, e_H), \quad \hat{\gamma}_b \equiv G_{ab}\hat{\gamma}^a. \tag{13}$$

The spinor connection is given by the following statement.

Statement 1

The spinor connection $\Gamma(x) = \gamma^i(x)\Gamma_i(x)$ in the homogeneous space M with invariant metric g_M reads

$$\Gamma(x) = \hat{\gamma}^a \left(\Gamma_a + \eta_a^\alpha(x, e_H)\Lambda_\alpha^s\right), \quad \Gamma_a = -\frac{1}{4}\Gamma_{ba}^d \hat{\gamma}^b \hat{\gamma}_d, \tag{14}$$

$$\Lambda_\alpha^s = -\frac{1}{8} G_{ac} C_{\alpha b}^a [\hat{\gamma}^b, \hat{\gamma}^c]. \tag{15}$$

A set of matrices Λ_α^s determines a spinor representation of the isotopy subgroup H in the space V_Ψ.

Statement 2

The matrices Λ_α^s are the generators of the isotropy subgroup H representation in the space V_Ψ.

Dirac equation in the homogeneous space

The generators of the transformation group G are determined by the left-invariant vector fields $\xi(g)$:

$$X_i(x) = \xi_i^b(x)\partial_{x^b}, \quad [X_i(x), X_j(x)] = C_{ij}^k X_k(x). \tag{16}$$

And they are symmetry operators of the Klein-Gordon equation (7) on the homogeneous space M. The symmetry operators of the Dirac equation have the form:

$$Y_i(x) = X_i(x) + \xi_i^\alpha(x, e_H)\Lambda_\alpha^s, \tag{17}$$
$$[Y_i(x), Y_j(x)] = C_{ij}^k Y_k(x).$$

We note that we consider symmetry operators generated by Killing vector fields on a homogeneous space. The spinor symmetry operators of the Dirac equation, which are generated by the Yano vector field and the Yano-Killing tensor field, will not be considered here.

λ–representation of Lie algebra

Let $\mathcal{O}_\lambda$ be a K-orbit of G, containing the covector λ. Introduce canonical Darbu coordinates $(p,q) \in P \times Q$, in terms of whitch the Kirillov form ω_λ takes its canonical form $\omega_\lambda = dp_a \wedge dq^a$. Define *a canonical embedding* $f : \mathcal{O}_\lambda \to \mathfrak{g}^*$, is uniquely defined by the function $f_X = f_X(p,q,\lambda)$, satisfying the system of equations

$$\{f_X, f_Y\}^{Lie} = f_{[X,Y]}, \quad f_X(0,0,\lambda) = \lambda(X), \quad X, Y \in \mathfrak{g}.$$

Let us now go over from the Lie algebra $\mathfrak{g}$ to its corresponding complex extension $\mathfrak{g}_\mathbb{C}$ consider a canonical embedding which is linear in the variables p:

$$f_X(q,p,\lambda) = \alpha^a_X(q)p_a + \chi_X(q,\lambda), \quad X \in \mathfrak{g}, \quad a = 1,\ldots,\dim Q. \tag{18}$$

The operators

$$\ell_X(q,\lambda) = if_X(\hat{q},\hat{p},\lambda) = \alpha^a_X(q)\partial_{q^a} + i\chi_X(q,\lambda) \tag{19}$$

realize an irreducible representation of the Lie algebra $\mathfrak{g}$ whitch acts on the functions from

$$\dim Q = \frac{1}{2}(\dim \mathfrak{g} - \operatorname{ind} \mathfrak{g}) \tag{20}$$

variables.

Noncommutative integration

We introduce the lifting of a λ-representation of a Lie algebra $\mathfrak{g}$ to a local representation of its Lie group G:

$$T^{\lambda}(g)\varphi(q) = \int D^{\lambda}_{q\overline{q}'}(g)\varphi(q')d\mu(q').$$

The ditributions $D^{\lambda}_{q\overline{q}'}(g)$ satisfy the overdetermined system of equations

$$[\eta_X(g) + \ell_X(q,\lambda)]D^{\lambda}_{q\overline{q}'}(g) = 0, \quad [\xi_X(g) - \overline{\ell^{\dagger}_X(q',\lambda)}]D^{\lambda}_{q\overline{q}'}(g) = 0, \tag{21}$$

The set of ditsributions $D^{\lambda}_{q\overline{q}'}(g)$ is complete and orthogonal. We seek basis of solutions for Klein-Gordon and Dirac equations in form

$$\psi_{\sigma}(x) = \int \varphi_{\sigma}(q')D^{\lambda}_{\overline{qq'}}(g(x,e_H)^{-1})d\mu(q'), \quad \sigma = (q,\lambda). \tag{22}$$

Reduced equations

1. Reduced Dirac equation:

$$\left(i\hat{\gamma}^{a}[\ell_a(q',\lambda)+\Gamma_a]-m\right)\varphi_\sigma(q')=0, \quad (23)$$

$$(\ell_\alpha(q',\lambda)+\Lambda^s_\alpha)\varphi_\sigma(q')=0. \quad (24)$$

2. Reduced Klein-Gordon equation:

$$\left(G^{ab}\left(\ell_a(q',\lambda)+\mathrm{Tr}(ad_b)\right)\ell_b(q',\lambda)+m^2\right)\varphi_\sigma(q')=0, \quad (25)$$

$$\ell_\alpha(q',\lambda)\varphi_\sigma(q')=0. \quad (26)$$

Solving first-order equations (24) and (26) we arrive at an equation with fewer variables $p < \dim M$. If $p = 0$ or $p = 1$, we say that the equation is non-commutative integrable.

Theorem 1

The Dirac and Klein-Gordon equations are non-commutative integrable on all four-dimensional homogeneous spaces with invariant metrics.

Vacuum expectation value of the scalar field EMT on a homogeneous space

We proceed with quantizing the scalar field on the homogeneous space M. We decompose the field operator $\hat{\varphi}(x)$ with respect to the basis of solutions of Klein-Gordon equation:

$$\hat{\varphi}(x) = \int d\mu(\sigma) \left[\psi_\sigma(x)\hat{a}_\sigma + \overline{\psi_\sigma(x)}\hat{a}^\dagger_\sigma\right], \tag{27}$$

where $d\mu(\sigma)$ is a measure for all quantum numbers and $\hat{a}^\dagger_\sigma$ and $\hat{a}_\sigma$ are the respective creation and annihilation operators. Covariant quantization is realized by imposing the commutation relations $[\hat{a}_\sigma, \hat{a}^\dagger_{\sigma'}] = \delta_{\sigma,\sigma'}$.The vacuum state corresponding to this quantization procedure is defined by the conditions

$$\hat{a}_\sigma|0\rangle = 0. \tag{28}$$

The vacuum expectation values of the EMT are then defined by an integral over all quantum numbers:

$$\langle \hat{T}_{ab}\rangle_0 = \int T_{ab}\{\psi_\sigma, \overline{\psi_\sigma}\} d\mu(\sigma), \tag{29}$$

where $T_{ab}\{\psi_\sigma, \overline{\psi_\sigma}\} = T_{ij}\{\psi_\sigma, \overline{\psi_\sigma}\}\eta^i_a(x)\eta^j_b(x)$,

$$T_{ij}\{\psi_\sigma, \overline{\psi_\sigma}\} = (1-2\zeta)\overline{\psi_{(,i}}\psi_{,j)} + \left(2\zeta - \frac{1}{2}\right) g_{ij}g^{kl}\overline{\psi_{,k}}\psi_{,l} - \tag{30}$$

$$-\zeta\left[(\nabla_i\nabla_j\overline{\psi})\psi + \overline{\psi}(\nabla_i\nabla_j)\psi\right] - \left(\zeta R_{ij} + \left(2\zeta - \frac{1}{2}\right) g_{ij}(m^2 + \zeta R)\right)\overline{\psi}\psi,$$

Tetrad components of vacuum expectation values

$$\langle \hat{T}_{ab}\rangle_0 = -\frac{1}{2\sqrt{|G|}}\int \overline{\varphi_\sigma(q')}\hat{K}_{ab}\varphi_\sigma(q')d\mu(q')d\mu(\sigma), \tag{31}$$

$$\hat{K}_{ab} = \frac{1}{2}\{\ell_a, \ell_b\} + \zeta R_{ab} + \tag{32}$$

$$+\left(2\zeta - \frac{1}{2}\right) G_{ab}G^{cd}\,\mathrm{Tr}(ad_c)\ell_d.$$

Vacuum expectation value of the spinor field EMT on a homogeneous space

Results for spinor field

$$\langle \hat{T}_{ab} \rangle_0 = \frac{i}{2} \int \overline{\varphi_\sigma(q')} \gamma_{(a} \ell_{b)} \varphi_\sigma(q') d\mu(q') d\mu(\sigma) + \tag{33}$$

$$+ \frac{i}{4} \int \left(\overline{\varphi_\sigma(q')} \gamma_{(a} \Gamma_{b)} \varphi_\sigma(q') - \overline{\Gamma_{(a} \varphi_\sigma(q')} \gamma_{b)} \varphi_\sigma(q') \right) d\mu(q') d\mu(\sigma).$$

- Right-invariant tetrads allows proceeding without using local coordinates on the homogeneous space; we can always pass to the standard EMT components at the end of the calculation. In tetrad components, we find expressions for vacuum expectation values of the EMT for scalar and spinor field that are defined by algebraic properties of the homogeneous space (such as a λ–representation of the Lie algebra).

- Divergences occur in calculating quantum expectation values over any state for operators (the EMT in particular) that are bilinear in the fields because bilinear operators contain products of operator-valued generalized functions. Hence, obtaining finite values of vacuum expectation values of the EMT requires using some procedure for removing the divergences.
- In the case where the space is homogeneous and isotropic, using the dimensional regularization method is efficient. Another way to regularize is by the method of splitting the arguments of field operators in the bilinear form of the EMT or zeta-regularization method. We note that although these regularization methods do not require calculating vacuum expectation values of the EMT, these last are also interesting because it is possible to eliminate the divergences directly in several cases (for example, using the n-wave regularization method).