Frame-dragging effect in the field of non rotating body due to unit gravimagnetic moment.
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Alexei A. Deriglazov (in collaboration with W. Guzman Ramirez)

Universidade Federal de Juiz de Fora, Brasil.

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(1) PLAN

1. Variational problem for a rotating body in general relativity (that is for Mathisson-Papapetrou-Tulczyjew-Dixon (MPTD) equations).

2. Three-dimensional acceleration in general relativity.

3. Unexpected behavior of MPTD-body in ultra-relativistic limit.

4. Modified equations (gravimagnetic body).

5. First post-Newtonian approximation: new $1/c^2$- corrections.

(2) Mathisson (1937), Papapetrou (1951), Tulczyjew (1959), Dixon (1964).

(MPTD)-equations in the form studied by Dixon:

$$\nabla P^\mu = -\frac{1}{4}R^\mu{}_{\nu\alpha\beta}\dot{x}^\nu S^{\alpha\beta}, \qquad \nabla S^{\mu\nu} = 2(P^\mu\dot{x}^\nu - P^\nu\dot{x}^\mu), \qquad S^{\mu\nu}P_\nu = 0,$$

They are assumed to describe evolution of a rotating body in gravitational field $g_{\mu\nu}$. In particular, they are used for computation of $1/c^2$-corrections to Lense-thirring and frame-dragging effects (measured during Stanford Gravity Probe B experiment, 2011) (Schiff 1960, Wald 1972, ...)

In these equations: $x^\mu(\tau)$ is "a representative point of the body", spin-tensor $S^{\mu\nu}(\tau)$ is associated with inner angular momentum, $P^\mu(\tau)$ is called "momentum".

From the beginning, they have been considered as a Hamiltonian-type system:

$$\dot{x}^\mu = \tfrac{\sqrt{-\dot{x}\mathcal{G}\dot{x}}}{-P^2}\mathcal{T}^{-1\mu}{}_\nu P^\nu, \quad P^2 + f(S^2) = -m^2c^2.$$

We realized this idea in explicit form, considering spin-tensor as a composite quantity: $S^{\mu\nu} = 2(\omega^\mu\pi^\nu - \omega^\nu\pi^\mu)$, where π^μ is conjugated momentum for the vector ω^μ - basic variable, taken for description of spin. In this case, there is the Lagrangian variational problem that implies expected equations of motion and constraints (AAD, Ramirez 2015).

(3) LAGRANGIAN FORMULATION OF MPTD-EQUATIONS

Configuration variables: position $x^\mu(\tau)$ and vector $\omega^\mu(\tau)$ attached to the point x^μ. Frenkel spin-tensor is composite quantity: $S^{\mu\nu} = 2(\omega^\mu\pi^\nu - \omega^\nu\pi^\mu)$, where π^μ is conjugated momentum for ω^μ.

Hanson-Regge type Lagrangian (parameter α determines spin: $S^2 = 8\alpha$):

$$L = -\frac{1}{\sqrt{2}}\sqrt{m^2c^2 - \frac{\alpha}{\omega^2}}\,\sqrt{\dot{x}N\dot{x} + \dot{\omega}N\dot{\omega} - \sqrt{[\dot{x}N\dot{x} + \dot{\omega}N\dot{\omega}]^2 - 4(\dot{x}N\dot{\omega})^2}}, \ \stackrel{\omega=0}{\longrightarrow}\ -mc\sqrt{\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$$

N is the projector on the plane orthogonal to ω^μ: $N_{\mu\nu} \equiv g_{\mu\nu} - \frac{\omega_\mu\omega_\nu}{\omega^2}$, then $N_{\mu\nu}\omega^\nu = 0$.

BMT-equations appear for the minimal spin-gravity interaction:

$$\eta_{\mu\nu} \to g_{\mu\nu}, \qquad \dot{\omega}^\mu \to \nabla\omega^\mu = \frac{d\omega^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\omega^\beta.$$

(4) DEFINITION OF THREE-ACCELERATION IN GENERAL RELATIVITY.

According to Landau-Lifshitz, time and space intervals between the events x^μ and $x^\mu + dx^\mu$ in curved space $g_{\mu\nu}$ are

$$dt = -\frac{g_{0\mu}dx^\mu}{c\sqrt{-g_{00}}}, \qquad dl^2 = \gamma_{ij}dx^i dx^j, \quad \text{where} \quad \gamma_{ij}(x^0, \mathbf{x}) \equiv \left(g_{ij} - \frac{g_{0i}g_{0j}}{g_{00}}\right) dx^i dx^j.$$

Then geodesic equation $\nabla \frac{dx^\mu}{ds} \equiv \frac{d^2x^\mu}{ds^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0$ implies the following variation rate of velocity components, $\frac{d^2}{dt^2}\mathbf{x}(t)$, in the direction of velocity:

$$(\mathbf{v}\gamma\frac{d\mathbf{v}}{dt}) = -\frac{\sqrt{c^2 - \mathbf{v}\gamma\mathbf{v}}}{c^2}(\mathbf{v}\gamma)_j\Gamma^j{}_{\mu\nu}(g)v^\mu v^\nu - \frac{(\mathbf{v}\gamma\mathbf{v})}{c^2}(\mathbf{v}\gamma)_p\tilde{\Gamma}^p{}_{jk}(\gamma)v^j v^k - \frac{(\mathbf{v}\gamma\mathbf{v})}{2c^2}(\mathbf{v}\partial_t\gamma\mathbf{v}).$$

This does not vanish as $|\mathbf{v}| \to c$. The reason is that the variation consist of three contributions: acceleration of the particle; variation of a basis in the passage from $\mathbf{x}$ to $\mathbf{x} + d\mathbf{x}$; and variation of the metric $\gamma(t)$ during the time interval dt. To obtain three-dimensional acceleration, we need to exclude the last two contributions:

$$a^i \stackrel{def}{=} \frac{dv^i}{dt} + \tilde{\Gamma}^i{}_{jk}(\gamma)v^j v^k + \frac{1}{2}(\mathbf{v}\partial_0\gamma\gamma^{-1})^i \equiv \nabla(\gamma)v^i + \frac{1}{2}(\mathbf{v}\partial_0\gamma\gamma^{-1})^i.$$

The definition guarantees that a point particle can not exceed the speed of light. Unfortunately, this does not happen for MPTD-particle.

(5) UNEXPECTED BEHAVIOR OF MPTD-PARTICLE AS $v \to c$.

A consistent manifestly relativistic and reparametrization-invariant equations can not contain too many velocities, since each v^μ implies $\frac{1}{\sqrt{c^2-v^2}}$ -factor:

Electrodynamics: $\frac{dv^\mu}{ds} = F^\mu{}_\nu v^\mu \quad \to \quad \mathbf{a}_{||} \sim \sqrt{c^2 - \mathbf{v}^2}$,

Geodesics: $\frac{dv^\mu}{ds} = -\Gamma^\mu{}_{\nu\alpha} v^\nu v^\alpha \to \quad \mathbf{a}_{||} \sim 1$,

Electrodynamics with non minimal interaction:

$$\frac{dv^\mu}{ds} = F^\mu{}_\nu v^\nu + (FF)_{\nu\alpha} v^\nu v^\alpha v^\mu \to \quad \mathbf{a}_{||} \sim \frac{1}{\sqrt{c^2-\mathbf{v}^2}}, \quad \dots \quad ,$$

and for MPTD-equations we have:

$$\nabla P^\mu = -\frac{1}{4} R^\mu{}_{\nu\alpha\beta} \dot{x}^\nu S^{\alpha\beta},$$

$$\nabla S^{\mu\nu} = 2(P^\mu \dot{x}^\nu - P^\nu \dot{x}^\mu), \qquad \to \frac{dv^\mu}{ds} \sim A^\mu{}_{\nu\alpha\beta} v^\nu v^\alpha v^\beta \to \mathbf{a}_{||} \sim \frac{1}{\sqrt{c^2-\mathbf{v}^2}}.$$

In the result, MPTD equations are not consistent in ultra-relativistic limit: longitudinal acceleration increases with velocity and diverges as $v \to c$.

(6) NON MINIMAL SPIN-GRAVITY INTERACTION: GRAVIMAGNETIC BODY

Our Lagrangian admits non minimal spin-gravity interaction through the gravimagnetic moment κ. This can be thought as a deformation of original metric (in spin-sector): $g^{\mu\nu} \to \sigma^{\mu\nu} = g^{\mu\nu} + \kappa R_\alpha{}^\mu{}_\beta{}^\nu \omega^\alpha \omega^\beta$. Denoting $K = (\sigma - \lambda^2 g)^{-1}$, the Lagrangian action is

$$S = -\int d\tau \sqrt{(mc)^2 - \frac{\alpha}{\omega^2}} \times \sqrt{-\dot{x}NK\sigma N\dot{x} - \nabla\omega NKN\nabla\omega + 2\lambda\dot{x}NKN\nabla\omega}.$$

The non minimal interaction modifies Hamiltonian (Khriplovich 1989):

$$H = P^2 + (mc)^2 \qquad \to \qquad H = P^2 + \kappa\left[\tfrac{1}{32}R_{\alpha\beta\mu\nu}S^{\alpha\beta}\right]S^{\mu\nu} + (mc)^2,$$

this can be compared with Hamiltonian of charged spinning particle with magnetic moment μ:

$$H = P^2 + \mu\left[\tfrac{e}{2c}F_{\mu\nu}\right]S^{\mu\nu} + (mc)^2.$$

The modified theory with $\kappa = 1$ (gravimagnetic body) is consistent in ultra-relativistic limit. It also differs from MPTD-theory for small velocities already in $1/c^2$ -approximation:

(7) IMPROVED MPTD-EQUATIONS: GRAVIMAGNETIC BODY

$$\left.\begin{array}{c}\text{MPTD equations}\\ \nabla P^\mu = -\frac{1}{4}R^\mu{}_{\nu\alpha\beta}\dot{x}^\nu S^{\alpha\beta},\\ \nabla S^{\mu\nu}=0.\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{Improved equations}\\ \nabla P_\mu = -\frac{1}{4}\theta_{\mu\nu}\dot{x}^\nu - \frac{\sqrt{-\dot{x}^2}}{32mc}(\nabla_\mu\theta_{\sigma\lambda})S^{\sigma\lambda},\\ \nabla S^{\mu\nu} = \frac{\sqrt{-\dot{x}^2}}{4mc}\theta^{[\mu}{}_\alpha S^{\nu]\alpha}.\end{array}\right.$$

Denoting $\theta_{\mu\nu} = R_{\mu\nu\alpha\beta}S^{\alpha\beta}$, the Lagrangian equations read ($1/c^2$-approximation):

$$\left.\begin{array}{c}\text{MPTD equations}\\ \nabla\left[\frac{T^\mu{}_\nu\dot{x}^\nu}{\sqrt{-\dot{x}(g+h)\dot{x}}}\right] = -\frac{1}{4mc}(\theta\dot{x})^\mu\\ \nabla S^{\mu\nu} = \frac{1}{4mc\sqrt{-\dot{x}(g+h)\dot{x}}}\dot{x}^{[\mu}(S\theta\dot{x})^{\nu]} \sim \frac{1}{\sqrt{-\dot{x}G\dot{x}}}.\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{Improved equations}\\ \nabla\left[\frac{\dot{x}^\mu}{\sqrt{-\dot{x}g\dot{x}}}\right] = -\frac{1}{4mc}(\theta\dot{x})^\mu - \frac{\sqrt{-\dot{x}g\dot{x}}}{32m^2c^2}\nabla^\mu(S\theta)\\ \nabla S^{\mu\nu} = \frac{\sqrt{-\dot{x}g\dot{x}}}{4mc}\theta^{[\mu}{}_\sigma S^{\nu]\sigma} \sim \sqrt{-\dot{x}g\dot{x}}.\end{array}\right.$$

Even in homogeneous field we have modified dynamics for both x and S. In the modified theory:

1. Longitudinal acceleration vanishes as $v \to c$.
2. Time interval and distance are unambiguously defined within the initial space-time metric $g_{\mu\nu}$.

Contrary to MPTD-equations, the modified theory is consistent with respect to the initial metric.

(8) EFFECTIVE $1/c^2$ -HAMILTONIAN

Consider gravimagnetic gyroscope in the metric of spherical body with mass M and angular momentum $\mathbf{J}$

$$ds^2 = \left(-1 + \frac{2GM}{c^2 r} - \frac{2G^2M^2}{c^4r^2}\right)(dx^0)^2 - 4G\frac{\epsilon_{ijk}J^jx^k}{c^3r^3}dx^0dx^i + \left(1 + \frac{2GM}{c^2r} + \frac{3G^2M^2}{2c^4r^2}\right)dx^idx^i.$$

With angular momentum $\mathbf{J}$ we assiciate the potential $\mathbf{A}_J$ of gravitomagnetic field $\mathbf{B}_J$:
$\mathbf{A}_J = \frac{2G}{c}[\mathbf{J} \times \frac{\mathbf{r}}{r^3}]$, $\qquad \mathbf{B}_J = [\boldsymbol{\nabla} \times \mathbf{A}_J]$;
and associate similar quantities with spin $\mathbf{S}$ of the particle:
$\mathbf{A}_S = \frac{M}{m}\frac{G}{c}[\mathbf{S} \times \frac{\mathbf{r}}{r^3}]$, $\qquad \mathbf{B}_S = [\boldsymbol{\nabla} \times \mathbf{A}_S]$.
With these notation, $1/c^2$ -Hamiltonian becomes similar to that of spinning particle in a magnetic field

$$H = \frac{c}{\sqrt{-g^{00}}}\sqrt{(mc)^2 + g^{ij}\Pi_i\Pi_j} + \frac{1}{2c}(\mathbf{B}_J + \mathbf{B}_S)\cdot\mathbf{S}, \qquad \text{where} \qquad \boldsymbol{\Pi} \equiv \mathbf{p} + \frac{m}{c}(\mathbf{A}_J + 2\mathbf{A}_S).$$

The approximate Hamiltonian (1) can be thought as describing a gyroscope orbiting in the field of Schwarzschild space-time $g_{\mu\nu}$, and interacting with the gravitomagnetic field $\mathbf{B}_J$ of central body and with fictitious gravitomagnetic field $\mathbf{B}_S$ due to spin of gyroscope. The only effect of non-minimsal interaction is the deformation of gravitomagnetic field of central body according to the rule

$$\mathbf{B}_J \to \mathbf{B}_J + \mathbf{B}_S.$$

(9) $1/c^2$-CORRECTIONS TO TRAJECTORY (LENSE-THIRRING)

Denoting $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$, total acceleration of gravimagnetic particle in $\frac{1}{c^2}$-approximation reads

$$\mathbf{a} = -\frac{MG}{r^2}\hat{\mathbf{r}} + \frac{4GM}{c^2 r^2}(\hat{\mathbf{r}}\cdot\mathbf{v})\mathbf{v} - \frac{GM}{c^2 r^2}v^2\hat{\mathbf{r}} + \frac{4G^2M^2}{c^2 r^3}\hat{\mathbf{r}} + \frac{1}{c}(\mathbf{B}_J + \mathbf{B}_S)\times\mathbf{v} + \frac{GM}{mc^2 r^3}\left[\mathbf{S}\times\mathbf{v} + 3(\mathbf{S}\cdot(\hat{\mathbf{r}}\times\mathbf{v}))\hat{\mathbf{r}}\right] - \frac{1}{2mc}\boldsymbol{\nabla}([\mathbf{B}_J + \mathbf{B}_S]\cdot\mathbf{S}).$$

The new term due to gravimagnetic moment is $-\frac{1}{2mc}\boldsymbol{\nabla}(\mathbf{B}_S\cdot\mathbf{S})$. As it should be expected, other terms coincide with those of known from analysis of MPTD equations Einstein 1915, de Sitter 1916, Thirring-Lense 1918, Schiff 1960, Wald 1972.

The first term represents the standard limit of Newtonian gravity and implies an elliptical orbit.

The next three terms represent an acceleration in the plane of orbit and are responsible for the precession of perihelia.

The term $\frac{1}{c}\mathbf{B}_J\times\mathbf{v}$ represents the acceleration due to Lense-Thirring rotation of central body, while the remaining terms describe influence of the gyroscopes spin on its trajectory.

The gravitational dipole-dipole contribution to Lense-Thirring effect $\frac{1}{2mc}\boldsymbol{\nabla}(\mathbf{B}_J\cdot\mathbf{S})$ has been computed by Wald.

The new contribution due to non minimal interaction, $\frac{1}{2mc}\boldsymbol{\nabla}(\mathbf{B}_S\cdot\mathbf{S})$, is similar to the Wald term.

(10) $1/c^2$-CORRECTIONS TO SPIN (FRAME DRAGGING)

In a co-moving frame, our effective Hamiltonian implies precession of spin $\frac{d\mathbf{S}}{dt} = [\mathbf{\Omega} \times \mathbf{S}]$ with angular velocity vector

$$\mathbf{\Omega} = \frac{3GM}{2c^2r^2}[\hat{\mathbf{r}} \times \mathbf{v}] + \frac{1}{2c}\mathbf{B}_J + \frac{1}{c}\mathbf{B}_S.$$

The first term represents geodetic precession of gyroscope.
The second term represents frame-dragging precession. The two terms are the same for both gravimagnetic and MPTD body. They have been computed by Schiff, and measured during Stanford Gravity Probe B experiment (2011).
The last term appears only for gravimagnetic particle and depends on gyroscopes spin $\mathbf{S}$. Hence, two gyroscopes with different magnitudes and directions of spin will precess around different rotation axes. Then the angle between their own rotation axes will change with time in Schwarzschild or Kerr space-time.
Comparing the last two terms, we conclude that precession of spin $\mathbf{S}$ due to gravimagnetic moment is equivalent to that of caused by rotation of central body with fictitious momentum

$$\mathbf{J}_{fict} = \frac{M}{m}\mathbf{S}.$$

Taking $\mathbf{J} = 0$ in the Hamiltonian, we conclude that, due to the term $\frac{1}{2c}\mathbf{B}_S \cdot \mathbf{S}$, spin of gravimagnetic particle will experience frame-dragging effect $\frac{1}{c}\mathbf{B}_S \times \mathbf{S}$ even in the field of non rotating central body.

(11) CONCLUSION

1. Rotating body with minimal spin-gravity interaction (MPTD-body) has unexpected ultra-relativistic limit.

2. Gravimagnetic body (non-minimal interaction through unit gravimagnetic moment) has consistent ultra-relativistic limit.

3. For small velocities, behavior of gravimagnetic body is different from MPTD-body already in the leading post-Newtonian approximation.

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