Pure radiation in space-time models that admit integration of the eikonal equation by the separation of variables method

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Assumption III

To construct integrable models, we will use spaces that allow the integration of the eikonal equation by the method of separation of variables.

3 - The space-time models with radiation

The energy-momentum tensor of pure radiation:

$$T_{ij} = \varepsilon \, L_i L_j, \tag{1}$$

where ε – energy density and L_i – wave vector of radiation.
The wave vector L_i is an isotropic vector and satisfies the norm condition:

$$g^{ij} L_i L_i = 0. \tag{2}$$

The conservation law of energy-momentum:

$$\nabla^i T_{ij} = 0. \tag{3}$$

where ∇_i is the covariant derivative.

The eikonal equation

$$g^{ij}\nabla_i S \nabla_j S = 0. \tag{4}$$

Definition

Let V_n be a n–dimensional Riemannian space with metric tensor g_{ij}. The eikonal equation (4) can be integrated by **complete separation of variables method** if co-ordinate set $\{u^i\}$ exists for which complete integral can be presented in the "separated"form:

$$S = \sum_{i=1}^{n} \phi_i(u^i, \lambda_1, ..., \lambda_n) \tag{5}$$

where $\lambda_1 ... \lambda_n$ – is the essential parameter.

Such spaces are called the **conformally Stäckel spaces**.

The Hamilton-Jacobi equation for test particle:

$$g^{ij}\nabla_i S \nabla_j S = m^2. \tag{6}$$

Spaces admitting complete separation of variables in (6) are called **the Stäckel spaces**.

Theorem

Let V_n be the Stäckel space. Then g_{ij} in privileged co-ordinate set can be shown in the form

$$g^{ij} = \sum_{\nu} (\Phi^{-1})_{n\nu}\, G^{ij}_{\nu}(x^{\nu}), \quad \Phi_{\mu\nu} = \Phi_{\mu\nu}\,(u^{\mu}),$$

$$G^{ij}_{\nu} = \delta^i_{\nu}\delta^j_{\nu}\varepsilon_{\nu}(u^{\nu}) + (\delta^i_{\nu}\delta^j_{p} + \delta^j_{\nu}\delta^i_{p})G^{\nu p}_{\nu} + \delta^i_{p}\delta^j_{q}G^{pq}_{\nu},$$

(in the last expression no summation over ν)

$$p, q = 1, \ldots N, \quad \nu, \mu = N+1, \ldots, n.$$

$\Phi_{\mu\nu}(u^{\mu})$ – *is called the Stäckel matrix.*

6 - Killing fields of Stäckel spaces

Geodesic equations of Stäckel spaces admit **the first integrals** that commutes pairwise with respect to the Poisson bracket:

$$X_{(\nu)} = X^{ij}_{(\nu)}\, p_i p_j, \qquad Y_{(q)} = Y^{i}_{(q)}\, p_i$$

$$X_{(\nu)(ij;k)} = Y_{(q)(i;j)} = 0$$

$$q = 1, ...N, \quad \nu, \mu = N+1, ..., n.$$

where p_i – momentum of test particle, the semicolon denotes the covariant derivative and the brackets denote symmetrization. Therefore ${Y_{(q)}}^i$, ${X_{(\nu)}}^{ij}$ are the components of **vector and tensor Killing fields** respectively.

7 - Covariant classification of Stäckel spaces

Definition

Pairwais commuting Killing vectors ${Y_{(q)}}^i$, where $q = 1, ...N$ and Killing tensors ${X_{(\nu)}}^{ij}$, where $\nu = N+1, ...n$ form a complete set of the type $(N.N_0)$, where

$$N_0 = N - rank|Y^i_{(p)} g_{ij} Y^j_{(q)}|$$

Theorem

A necessary and sufficient geometrical criterion of a Stäckel space is the presence of a complete set of the type $(N.N_0)$.

Then the Hamilton-Jacobi equation and eikonal equation can be integrated by the complete separation of variables method if and only if **the complete set of the first integrals** exists.

For space-times admitting complete separation of variables in eikonal equation, using the norm condition

$$g^{ij} L_i L_i = 0, \tag{7}$$

and finding the integrals of the conservation law of energy-momentum

$$\nabla^i T_{ij} = 0, \tag{8}$$

we can obtain functional expressions for **the wave vector** and **the energy density of radiation**.

Below are listed the solutions for all types of this space-times without the use of field equations of the concrete theory of gravitation.

9 - Conformally Stäckel space-times (3.1) type

Conformally Stäckel space-times (3.1) type admits 3 commuting Killing vectors $Y^i_{(p)}$ $(p = 1, 3)$, but $rank\,|Y^i_{(p)} g_{ij} Y^j_{(q)}| = 2$. In a privileged coordinate system the metric of a conformally Stäckel space-times (3.1) type can be written in the following form, where the variable x^0 is a null ("wave-like") variable:

$$g^{ij} = \frac{1}{\Delta} \begin{pmatrix} 0 & 1 & a_0 & b_0 \\ 1 & 0 & 0 & 0 \\ a_0 & 0 & c_0 & f_0 \\ b_0 & 0 & f_0 & d_0 \end{pmatrix}, \qquad (9)$$

where $\Delta = \Delta(x^0, x^1, x^2, x^3)$ and a_0, b_0, c_0, d_0, f_0 are functions of a variable x^0.

The wave vector of the radiation has the form:

$$L_0 = L_0(x^0), \quad L_1 = \alpha, \quad L_2 = \beta, \quad L_3 = \gamma,$$
$$\alpha, \beta, \gamma - const.$$

The normalization condition and the conservation equations reduce to:

$$\beta^2 c_0 + 2\beta\gamma f_0 + \gamma^2 d_0 + 2(\alpha + \beta a_0 + \gamma b_0) L_0 = 0, \tag{10}$$

$$(\alpha + \beta a_0 + \gamma b_0) P_{,0} + L_0 P_{,1} + (a_0 L_0 + \beta c_0 + \gamma f_0) P_{,2}$$
$$+ (b_0 L_0 + \beta f_0 + \gamma d_0) P_{,3} + 2\beta a_0' + 2\gamma b_0' = 0. \tag{11}$$

From equations (10)-(11) we can obtain functional expressions for the wave vector and the energy density of radiation.

11 - Conformally Stäckel space-times (3.1) type

From the normalization condition and the conservation equations we obtain:

$$L_i = \left(L_0(x^0), \alpha, \beta, \gamma\right), \quad \alpha, \beta, \gamma - const, \tag{12}$$

$$L_0 = \frac{-(\beta^2 c_0 + 2\beta\gamma f_0 + \gamma^2 d_0)}{2\,(\alpha + \beta a_0 + \gamma b_0)}. \tag{13}$$

For the energy density of radiation we obtain:

$$\varepsilon = F(X, Y, Z)\Delta\sqrt{-\det g^{ij}}/(\alpha + \beta a_0 + \gamma b_0), \tag{14}$$

$$\begin{aligned} X &= x^1 - \int \frac{L_0}{(\alpha + \beta a_0 + \gamma b_0)}\, dx^0, \\ Y &= x^2 - \int \frac{(a_0 L_0 + \beta c_0 + \gamma f_0)}{(\alpha + \beta a_0 + \gamma b_0)}\, dx^0, \\ Z &= x^3 - \int \frac{(b_0 L_0 + \beta f_0 + \gamma d_0)}{(\alpha + \beta a_0 + \gamma b_0)}\, dx^0, \end{aligned}$$

where $F(X, Y, Z)$ is an arbitrary function of its variables.

Conformally Stäckel space-times (2.1) type admits two commuting Killing vectors $Y^i_{(p)}$ $(p = 1, 2)$, but $rank\,|Y^i_{(p)} g_{ij} Y^j_{(q)}| = 1$. In a privileged coordinate system the metric of a conformally Stäckel space-times (2.1) type can be written in the following form, where x^1 is a null ("wave-like") variable:

$$g^{ij} = \frac{1}{\Delta}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & f_1(x^1) & 1 \\ 0 & f_1(x^1) & A & B \\ 0 & 1 & B & C \end{pmatrix}, \qquad (15)$$

$$\Delta = \Delta(x^0, x^1, x^2, x^3), \quad A = a_0(x^0) + a_1(x^1),$$
$$B = b_0(x^0) + b_1(x^1), \quad C = c_0(x^0) + c_1(x^1).$$

The wave vector of radiation L_i has the form:

$$L_0 = L_0(x^0), \quad L_1 = L_1(x^1),$$
$$L_2 = \alpha, \quad L_3 = \beta, \quad \alpha, \beta, \gamma - const. \qquad (16)$$

From the norm condition one can obtain:

$$\gamma = {L_0}^2 + \alpha^2 a_0 + 2\alpha\beta b_0 + \beta^2 c_0, \tag{17}$$

$$-\gamma = 2(\alpha f_1 + \beta)L_1 + \alpha^2 a_1 + 2\alpha\beta b_1 + \beta^2 c_1. \tag{18}$$

From the conservation law one can obtain the equation for the energy density of radiation:

$$L_0 P_{,0} + (\alpha f_1 + \beta)P_{,1} + (\alpha A + \beta B + f_1 L_1)P_{,2}$$

$$+ (\alpha B + \beta C + L_1)P_{,3} + 2\alpha f_1' + 2{L_0}' = 0. \tag{19}$$

From the system of equations (18)–(19) we can obtain functional expressions for the wave vector and the radiation energy density.

The wave vector of radiation has the form:

$$L_i = \left(L_0(x^0), L_1(x^1), \alpha, \beta\right), \quad \alpha, \beta, \gamma - const,$$

$$L_0 = \sqrt{\gamma - \alpha^2 a_0 - 2\alpha\beta b_0 - \beta^2 c_0}, \tag{20}$$

$$L_1 = (-\gamma - \alpha^2 a_1 - 2\alpha\beta b_1 - \beta^2 c_1)/\left(2\left(\alpha f_1 + \beta\right)\right).$$

The radiation energy density has the form:

$$\varepsilon = \frac{F(X, Y, Z)\Delta\sqrt{-\det g^{ij}}}{L_0(\alpha f_1 + \beta)}, \tag{21}$$

$$X = \int \frac{dx^0}{L_0} - \int \frac{dx^1}{(\alpha f_1 + \beta)},$$

$$Y = x^2 - \int \frac{(\alpha a_0 + \beta b_0)}{L_0}\, dx^0 - \int \frac{(\alpha a_1 + \beta b_1 + f_1 L_1)}{\alpha f_1 + \beta}\, dx^1,$$

$$Z = x^3 - \int \frac{(\alpha b_0 + \beta c_0)}{L_0}\, dx^0 - \int \frac{(\alpha b_1 + \beta c_1 + L_1)}{\alpha f_1 + \beta}\, dx^1,$$

where $F(X, Y, Z)$ is an arbitrary function of its variables.

Conformally Stäckel space-times (1.1) type admits one Killing vector. In a privileged coordinate system the metric can be written in the following form, where x^1 is a null ("wave-like") variable:

$$g^{ij} = \frac{1}{\Delta} \begin{pmatrix} \Omega & V^1 & 0 & 0 \\ V^1 & 0 & 0 & 0 \\ 0 & 0 & V^2 & 0 \\ 0 & 0 & 0 & V^3 \end{pmatrix}, \qquad (22)$$

$$\Delta = \Delta(x^0, x^1, x^2, x^3),$$
$$V^1 = t_2(x^2) - t_3(x^3), \quad V^2 = t_3(x^3) - t_1(x^1),$$
$$V^3 = t_1(x^1) - t_2(x^2), \quad \Omega = \omega_\mu(x^\mu) V^\mu, \quad \mu, \nu = 1...3.$$

The wave vector of radiation has the following "separated"form:

$$L_i = \Big(\alpha, L_1(x^1), L_2(x^2), L_3(x^3)\Big), \quad \alpha - const. \qquad (23)$$

From norm condition we have:

$$\alpha V^1(2L_1 + \alpha\omega_1) + V^2({L_2}^2 + \alpha^2\omega_2) + V^3({L_3}^2 + \alpha^2\omega_3) = 0. \tag{24}$$

From equations of conservation law we have:

$$(V^1 L_1 + \alpha\Omega)P_{,0} + \alpha V^1 P_{,1} + L_2 V^2 P_{,2} + L_3 V^3 P_{,3} + 2V^2 L_2' + 2V^3 L_3' = 0. \tag{25}$$

From this system of equations we obtain functional expressions for the wave vector and the radiation energy density through the metric.

The wave vector of radiation has the form:

$$L_0 = \alpha, \quad L_1 = \frac{1}{2\alpha}(\beta t_1 - \alpha^2\omega_1 + \gamma), \quad \alpha, \beta, \gamma - const,$$

$$L_2 = \sqrt{\beta t_2 - \alpha^2\omega_2 + \gamma}, \quad L_3 = \sqrt{\beta t_3 - \alpha^2\omega_3 + \gamma}. \tag{26}$$

The energy density of the radiation has the form:

$$\varepsilon = F(X, Y, Z)\,\Delta\sqrt{-\det g^{ij}}/(L_2\, L_3), \tag{27}$$

$$X = x^0 - \frac{1}{\alpha}\int(L_1 + \alpha\omega_1)\,dx^1 - \alpha\left(\int\frac{\omega_2}{L_2}\,dx^2 + \int\frac{\omega_3}{L_3}\,dx^3\right),$$

$$Y = -\frac{1}{\alpha}\int t_1\,dx^1 + \int\frac{t_2}{L_2}\,dx^2 + \int\frac{t_3}{L_3}\,dx^3,$$

$$Z = \frac{x^1}{\alpha} + \int\frac{dx^2}{L_2} + \int\frac{dx^3}{L_3},$$

where $F(X, Y, Z)$ is an arbitrary function of its variables.

18 - Conclusion

For space-time models with pure radiation we suggest a method for obtaining analytical solutions in **any metric theories of gravity** based on the use of coordinate systems that admit separation of variables in the eikonal equation.

The method is based on **integrating the energy-momentum conservation equations**.

In the report we present **a classification of the solutions** of the energy-momentum conservation equations for all types of space-times that allow the separation of variables in the eikonal equation.

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