Models of the general scalar-tensor theory of gravitation with radiation, allowing the separation of variables in the Hamilton-Jacobi equation for test particles

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General form of Lagrangian in scalar-tensor theories of gravitation

$$L = \frac{1}{16\pi}\sqrt{-g}\left[\phi R - \frac{\omega(\phi)}{\phi}\nabla_i\phi\nabla^i\phi - 2\Lambda(\phi)\right] + L_{matter}$$

Two most popular limits:

1) Brance-Dicke-Jordan theory

$$\Lambda \rightarrow 0, \quad \omega \rightarrow const,$$

2) General relativity

$$\Lambda \rightarrow const, \quad \omega \rightarrow 0.$$

In generalized theories arbitrary functions $\Lambda(\phi)$, $\omega(\phi)$ are used.

The main problem: don‘t fix that functions!
Try to find them!

The field equations for the presented Lagrangian

$$(2\omega(\phi)+3)\nabla^k\nabla_k\phi+\omega'(\phi)\,\nabla^k\phi\nabla_k\phi+4\Lambda(\phi)-2\phi\Lambda'(\phi)=8\pi T,$$

$$\Theta_{ij}(\phi,g_{kl})=8\pi T_{ij},$$

where

$$\Theta_{ij}(\phi,g_{kl})=$$

$$\phi\,G_{ij}+\left(\nabla^k\nabla_k\phi+\frac{1}{2}\frac{\omega(\phi)}{\phi}\nabla^k\phi\nabla_k\phi+\Lambda(\phi)\right)g_{ij}-\nabla_i\nabla_j\phi-\frac{\omega(\phi)}{\phi}\nabla_i\phi\nabla_j\phi.$$

Let us consieder the pure radiation as a matter

$$T_{ij}=\varepsilon\,k_ik_j,\qquad k^ik_i=0,\qquad \nabla^iT_{ij}=0,$$

and a space-time with the Complete separation of variables in Hamilton-Jacobi equation!

Complete separation of variables in Hamilton-Jacobi equation is connected with integrals of motion

$$\underset{p}{Y} = \underset{p}{Y}{}^{i} p_i, \qquad \underset{\nu}{X} = \underset{\nu}{X}{}^{ij} p_i p_j,$$

where Y^i and X^{ij} — the Killing vectors and tensors of complete set:

N commuting Killing vectors ${Y_p}^i$ and $n - N$ commuting Killing tensors ${X_\nu}^{ij}$ form **complete set** *of type $(N.N_0)$, if they are indepedent and* $rank|| {Y_p}^i Y_{q\,i}|| = N - N_0$

The ability of existing of priviliged coordinate system, in which complete separation in Hamilton-Jacobi equation can be reached, is a restriction on the space. Such spaces are called **Stackel spaces**.

In a priviliged coordinate system:

1) the coordinate x^a corresponding to the Killing vector Y_a of the complete set is ignored: $\partial g_{ij}/\partial x^a = 0$;

2) the vawe vector has a separated form: $k_i = k_i(x^i)$.

If the Killing vector is izotropical, the corresponding coordinate is a null (vawe-like) variable, this leads to the zero on the main diagonal of the contravariant metric tensor.

Stackel space of the type (2.1) in the priviliged coordinate system has folloving:

1) 2 killing vectors of complete set

$$Y_1 = (0, 0, 1, 0), \qquad Y_2 = (0, 0, 0, 1),$$

2) a separated form of a vawe-vector

$$k = (k_0(x^0), k_1(x^1), \delta, \gamma), \qquad \delta, \gamma = const,$$

3) a contravariant metric tensor in the view

$$g^{ij} = \frac{1}{\Delta} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & f_1 & 1 \\ 0 & f_1 & A & B \\ 0 & 1 & B & C \end{pmatrix}$$

$$\Delta = t_0(x^0) + t_1(x^1), \; A = a_0(x^0) + a_1(x^1), \; B = b_0(x^0) + b_1(x^1),$$
$$C = c_0(x^0) + c_1(x^1).$$

Similarly to known fact about separation variables in the Klein-Gordon equation, we will suppose a separated form of scalar field

$$\phi = \phi_0(x^0)\phi_1(x^1)$$

$$(or \quad \Phi = \Phi_0 + \Phi_1 \quad for \quad \Phi = \ln \phi)$$

The field equations in the priviliged coordinate system

$$\Lambda(\phi) = -\frac{\phi}{4\Delta}\{(d_{,0}+\Phi_{,0})^2 + (3+2\omega(\phi))\Phi_{,0}{}^2 + 4(d_{,00}+\Phi_{,00})\}$$

$$2d_{,0}d_{,1} + (d_{,0}+\Phi_{,0})(d_{,1}+\Phi_{,1}) - (2\omega(\phi)+3)\Phi_{,0}\Phi_{,1} - \frac{3}{2}\frac{a_0{}'a_1{}'}{A^2} = 0$$

$$\epsilon k_1{}^2 = \frac{\phi}{8}\left\{\frac{a_1{}''}{A} - \frac{3}{2}\frac{a_1{}'^2}{A^2} - \frac{b_0{}'^2}{A} - \frac{a_0{}'c_0{}'}{2A} + c_0{}'' + c_0{}'(d_{,0}+\Phi_{,0}) + \right.$$

$$\left.(d_{,1}+\Phi_{,1})^2 - 2(d_{,11}+\Phi_{,11}) - (3+2\omega(\phi))\Phi_{,1}{}^2\right\}$$

$$d_{,00}+\Phi_{,00} = \frac{1}{2}\left\{(d_{,0}+\Phi_{,0})^2 - (3+2\omega(\phi))\Phi_{,0}{}^2 - \frac{a_0{}'}{A}(d_{,0}+\Phi_{,0})\right\}$$

$$a_0{}'' = -(d_{,0}+\Phi_{,0})a_0{}' + \frac{3}{2}\frac{a_0{}'^2}{A}$$

$$b_0{}'' = -(d_{,0}+\Phi_{,0})b_0{}' + \frac{3}{2}\frac{a_0{}'b_0{}'}{A}$$

(the scalar equation is satisfied identically)

Solution S1

$$k_i = (0, k_1, 0, 0), \qquad \phi = {x^0}^{2/\lambda}\phi_1, \qquad \alpha, \beta, \lambda = const, \qquad \lambda \neq 0,$$

$$g^{ij} = \frac{1}{\beta{\phi_0}^{-\lambda} + \alpha{\phi_1}^{\lambda}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & a_1 & b_1 \\ 0 & 1 & b_1 & C \end{pmatrix},$$

$$\Lambda(\phi) = -\frac{\phi(\beta(\lambda - 1)(3\lambda - 2) - \alpha(\lambda - 2)\phi^{\lambda})}{\lambda^2(\beta + \alpha\phi^{\lambda})^2}, \qquad \omega(\phi) = -1 + \frac{\lambda}{2} - \beta\,\frac{\lambda}{2}\,\frac{2\beta + \alpha(2 + 3\lambda)\phi^{\lambda}}{(\beta + \alpha\phi^{\lambda})^2}$$

$$\epsilon{k_1}^2 = \frac{\phi}{2}\left(\frac{{a_1}''}{a_1} - \frac{3}{2}\frac{{a_1}'^2}{{a_1}^2} + {c_0}'' + \frac{2}{\lambda}\frac{{c_0}'}{x^0} + \frac{\beta + (\lambda + 1)\alpha\phi^{\lambda}}{\beta + \alpha\phi^{\lambda}}\left((\lambda + 2)\frac{{\phi_1}'^2}{{\phi_1}^2} - 2\frac{{\phi_1}''}{\phi_1}\right) - \frac{2\beta}{\beta + \alpha\phi^{\lambda}}\frac{{c_0}'}{x^0}\right)$$

$$R = -6\beta\frac{2\beta + 3\alpha\phi^{\lambda}}{(\beta + \alpha\phi^{\lambda})^3},$$

$$C_{0101} = \frac{\beta + \alpha\phi^{\lambda}}{8{a_1}^2{x^0}^2}(3{a_1}'^2 - 2a_1{a_1}'' + 2{a_1}^2{c_0}''), \qquad C_{1212} = -C_{0101}/a_1.$$

Solution S2

$$k_i = (0, k_1, 0, 0), \qquad \phi = e^{\alpha x^0} e^{t_1} \quad (\Phi = \alpha x^0 + t_1),$$

$$g^{ij} = \frac{1}{\alpha x^0 + t_1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & a_1 & b_1 \\ 0 & 1 & b_1 & C \end{pmatrix},$$

$$\Lambda(\phi) = -\frac{\alpha^2 \phi}{2} \left(1 + \frac{2}{\Phi}\right) \frac{1}{\Phi}, \qquad \omega(\phi) = -1 + \frac{3}{2} \Phi^{-2} + \Phi^{-1},$$

$$\epsilon {k_1}^2 = \frac{\phi}{2} \left(\frac{{a_1}''}{a_1} - \frac{3}{2} \frac{{a_1}'^2}{{a_1}^2} + {c_0}'' + \left(1 + \frac{1}{\Phi}\right) \left(\alpha {c_0}' - \frac{{t_1}''}{{a_1}^2}\right) \right).$$

$$R = \frac{3\alpha^2}{2\Phi^3}, \qquad C_{0101} = \frac{\Phi}{8{a_1}^2}(3{a_1}'^2 - 2a_1{a_1}'' + 2{a_1}^2{c_0}''), \qquad C_{1212} = -C_{0101}/a_1.$$

Different cases of fucntions $\omega(\phi), \Lambda(\phi)$

1. Non-constant ω

$$\Lambda(\phi) = -\frac{\phi(\beta(\lambda-1)(3\lambda-2) - \alpha(\lambda-2)\phi^{\lambda})}{\lambda^2(\beta+\alpha\phi^{\lambda})^2}, \qquad \omega(\phi) = -1 + \frac{\lambda}{2} - \beta\,\frac{\lambda}{2}\,\frac{2\beta + \alpha(2+3\lambda)\phi^{\lambda}}{(\beta+\alpha\phi^{\lambda})^2}$$

$$\Lambda(\phi) = -\frac{\alpha^2\phi}{2}\left(1 + \frac{2}{\Phi}\right)\frac{1}{\Phi}, \qquad \omega(\phi) = -1 + \frac{3}{2}\,\Phi^{-2} + \Phi^{-1},$$

2. Constant ω

$$\omega = const, \qquad \Lambda \sim \phi,$$

$$\omega = 0, \qquad \Lambda \sim \frac{1}{\phi},$$

$$\omega = -\frac{3}{2}, \qquad \Lambda = 0.$$

$$\alpha, \beta, \lambda - const, \qquad \lambda \neq 0, \qquad \Phi = \ln\phi.$$

Conclusion

This work demonstrates the ability of separation of variables and exact solutions obtaining for generalized scalar-tensor theory of gravity with spaces with complete separation of variables in Hamilton-Jacobi equation as model of space-time.

11 exact solutions for space of type (2.1) are obtained.

There are 5 different forms of depedency $\Lambda(\phi)$, $\omega(\phi)$ in found solutions.