Cosmological aspects of the Eisenhart lift

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Plan

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I. Eisenhart lift

A dynamical system with d degrees of freedom $(i = 1, \dots, d)$

$$L = \frac{1}{2}\dot{x}_i\dot{x}_i - U(t,x)$$

is described by null geodesics in $(d+2)$–dimensional spacetime of Lorentzian signature $(y^\mu = (t, v, x_i))$ (L. Eisenhart, 1929)

$$g_{\mu\nu}(y)dy^\mu dy^\nu = -2U(t,x)dt^2 - dtdv + dx_i dx_i$$

Computing the Christoffel symbols and analyzing the geodesic equations one finds

$$\frac{d^2t}{d\tau^2} = 0 \quad \Rightarrow \quad \frac{dt}{d\tau} = \kappa$$

$$\frac{d^2x_i}{dt^2} + \partial_i U(t,x) = 0$$

$$\frac{dv}{dt} = \frac{dx_i}{dt}\frac{dx_i}{dt} - 2U - \frac{\epsilon}{\kappa^2}$$

where $\epsilon = 0$ is for null geodesics and $\epsilon = -1$ for time–like geodesics. The original Newtonian mechanics is recovered by implementing the null reduction along v.

Covariantly constant null Killing vector field

$$v' = v + a \quad \Rightarrow \quad \xi^\mu \partial_\mu = \partial_v, \qquad \nabla_\mu \xi_\nu = 0, \qquad \xi^2 = 0$$

gives rise to the energy–momentum tensor

$$T_{\mu\nu} = \Omega(t,x)\xi_\mu \xi_\nu, \qquad T^\mu{}_\mu = 0, \qquad \nabla^\mu T_{\mu\nu} = 0$$

The Ricci tensor and the scalar curvature ($\Delta = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}$)

$$R_{tt} = \Delta U, \qquad R = 0$$

The Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$$

reduce to the Poisson equation which underlies the Newtonian gravity

$$\Delta U = 2\pi\Omega$$

Eisenhart lift

The Eisenhart metric is a particular case of the Brinkmann metric or Pp-wave (H.W. Brinkmann, 1925)

Consider a metric written in the Kerr-Schild form

$$g_{\mu\nu}(y) = \eta_{\mu\nu} + l_\mu l_\nu F(y)$$

Impose the simplifying conditions

$$l^2 = 0, \qquad l^\mu \partial_\mu F(y) = 0$$

The vacuum Einstein equations hold provided $F(y)$ obeys the wave equation

$$R_{\mu\nu} = -\frac{1}{2} l_\mu l_\nu \Box F(y)$$

Choosing for definiteness $l_\mu = (1, -1, 0, 0)$ and denoting $y^\mu = (T, V, x_1, x_2)$, $t = T - V$, $v = T + V$, one gets the Brinkmann metric or Pp-wave

$$ds^2 = dtdv - d\vec{x}^2 + F(u, \vec{x})dt^2, \qquad \Delta F(t, \vec{x}) = 0$$

Consider the isotropic oscillator driven by the $d=1$ conformal mode

$$\rho(t)^2\frac{d}{dt}\left(\rho(t)^2\frac{d}{dt}x_i(t)\right)+\gamma^2 x_i(t)=0, \qquad \frac{d^2\rho(t)}{dt^2}=\frac{\gamma^2}{\rho(t)^3}-\frac{\rho(t)}{L^2}$$

where γ and L are positive constants. This system exhibits the conformal Newton–Hooke symmetry (S. Fedoruk, E. Ivanov, J. Lukierski)
Changing the evolution parameter

$$\rho(t)^2\frac{d}{dt}=\frac{d}{d\varphi}, \qquad \frac{d\varphi}{dt}=\frac{1}{\rho^2}$$

one concludes that the x–particle moves along the ellipse

$$x_i(t)=\mu_i\cos\left(\gamma\varphi(t)\right)+\nu_i\sin\left(\gamma\varphi(t)\right)$$

while $\rho(t)$ is an oscillating function which determines the angular velocity.

Can $\rho(t)$ be interpreted as a cosmic scale factor in the Eisenhart metric?

The first integral of the $d=1$ conformal mechanics resembles the Friedmann equation for the radiation dominated Universe

$$\dot{\rho}^2=-\frac{\gamma^2}{\rho^2}-\frac{\rho^2}{L^2}+E, \qquad \dot{a}^2=\frac{C}{a^2}+\frac{\Lambda}{3}a^2-k$$

III. Eisenhart metric with cosmic scale factor

Consider the Eisenhart metric associated with the isotropic oscillator

$$ds^2 = -\frac{\gamma^2 x_i x_i}{a(t)^2} dt^2 - dt dv + a(t)^2 dx_i dx_i$$

where $a(t)$ is a cosmic scale factor and γ is a constant.

Choosing the energy–momentum tensor as above and setting $\Omega = \Omega(t)$, the Einstein equations reduce to the Ermakov equation for the cosmic scale factor

$$\ddot{a} + \Omega(t) a = \frac{\gamma^2}{a^3}$$

Introducing the conformal time

$$a^2(t)\frac{d}{dt} = \frac{d}{d\eta}, \qquad \eta(t) = \int_{t_0}^{t} \frac{d\tilde{t}}{a^2(\tilde{t})}$$

one finds

$$ds^2 = a^2(\eta)\left(-\gamma^2 x_i x_i d\eta^2 - d\eta dv + dx_i dx_i\right)$$

This is an analog of the flat (k=0) FLRW cosmological model in which the Minkowski metric is changed by the simplest Pp–wave.

IV. Ermakov equation

The Ermakov equation (1880)

$$\ddot{a} + \Omega^2(t)a = \frac{1}{a^3}$$

gives a clue to the construction of the Lewis invariant invariant (H.R. Lewis, 1968)

$$I = \frac{x^2}{a^2} + (\dot{x}a - x\dot{a})^2$$

for the time–dependent harmonic oscillator

$$\ddot{x} + \Omega(t)^2 x = 0$$

Given the general solutions to the time–dependent harmonic oscillator and the Ermakov equation, the Lewis invariant is conserved over time

$$\frac{dI}{dt} = 0$$

Switching to the conformal time, one can solve the geodesic equations by quadrature

$$\tau = \kappa \int_{\eta_0}^{\eta} a^2(\tilde{\eta}) d\tilde{\eta} + \tau_0,$$

$$x_i(\eta) = \alpha_i \cos(\gamma\eta) + \beta_i \sin(\gamma\eta),$$

$$v(\eta) = v_0 - \frac{1}{2}(\alpha^2 - \beta^2)\gamma \sin(2\gamma\eta) + \alpha\beta\gamma \cos(2\gamma\eta) - \epsilon\kappa^2 \int_{\eta_0}^{\eta} a^2(\tilde{\eta}) d\tilde{\eta},$$

where τ_0, κ, α_i, β_i, v_0 are constants of integration and $\alpha^2 = \alpha_i\alpha_i$, $\alpha\beta = \alpha_i\beta_i$.

A particular solution for null geodesics

$$x_i(\eta) = 0, \qquad v(\eta) = v_0$$

implies that the metric is given in a reference frame comoving with a light signal which travels along the v–axis.

Null geodesics in Ermakov cosmoi

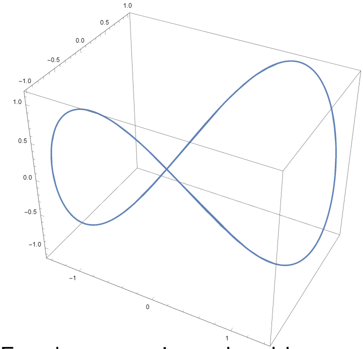

Figure : Null geodesics in Ermakov cosmoi are closed loops which wrap around elliptic cylinders. The graph lies in three–dimensional space with (x_1, x_2, v) parametrizing the two horizontal and the vertical axes, respectively.

Time–like geodesics in Ermakov cosmoi

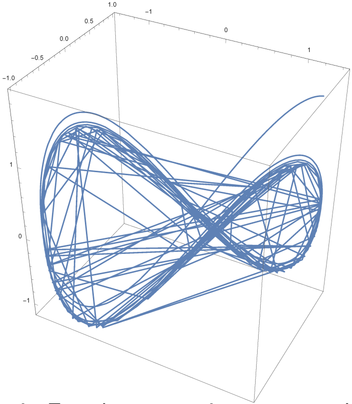

Figure : Time–like geodesics in Ermakov cosmoi wrap around elliptic cylinders. They remain in a compact region of space for some time before escaping. The graph is given in three–dimensional space, (x_1, x_2, v) parametrizing the two horizontal and one vertical axes, respectively.

Little is known about analytic solutions to the Ermakov equation. Assuming that the energy density $\Omega(t)$ decreases with time, one has

$$\Omega(t) = \frac{h^{n-2}}{4\nu^2 t^n}, \qquad \nu = \frac{1}{2-n}, \qquad n \neq 2$$

$$a(t) = \sqrt{\pi\gamma\nu t(J_\nu^2(\lambda) + Y_\nu^2(\lambda))}, \qquad \lambda = \left(\frac{t}{h}\right)^{\frac{1}{2\nu}}$$

where n is a rational number such that ν is positive, h is a positive constant, and J_ν, Y_ν are Bessel functions.

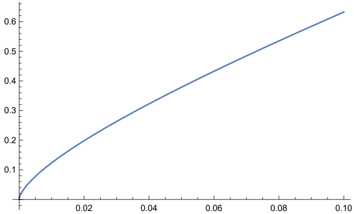

Figure : A graph of the cosmic scale factor in the cosmological model at $n = 3$.

Although the cosmic scale factors in the Ermakov cosmology look quite reasonable, the geodesic motion is apparently unrealistic. A possible cure is to generalize the construction to the case of time dependent γ

$$\ddot{a} + \Omega(t) a = \frac{\gamma(t)^2}{a^3}$$

This can be viewed as an algebraic equation to fix $\gamma(t)$ in terms of $a(t)$ and $\Omega(t)$.

In this way one can model a reasonable geodesic behavior in the generalized Ermakov cosmoi by properly choosing the cosmic scale factor and the energy density. However, as the associated geodesic equations involve a time dependent oscillator with frequency $\gamma(t)^2$, finding an analytic solution may turn out to be rather complicated.

Thanks for your attention!