First order derivatively coupled scalar-tensor theory and non-singular cosmology

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Introduction

Detection of the binary neutron star merger GW170817 by LIGO-VIRGO collaboration followed by observations of its electromagnetic counterparts showed that the velocity of gravitational waves is equal to the velocity of light with 10^{-15} accuracy.

This discovery already had serious impact on the modified gravity theories partly refuting the most p pular Horndeski (covariant Galileon) and beyond Horndeski models.

At the moment, the restrictions imposed by the speed of gravitational waves are more stringent than the traditional cosmological bounds of these theories.

Therefore, one is urged to revise the list of existing models and look for new ones that could pass this test.

We consider the action with two independent couplings of the derivatives of the scalar field $\phi_\mu \equiv \phi_{,\mu}$ to the Ricci tensor and the Ricci scalar
(Amendola:1993,Capozziello:1999,Sushkov:2009,Granda:2010)

$$S = \int d^4x\sqrt{-g}\,[R - (g_{\mu\nu} + \kappa_1 g_{\mu\nu}R + \kappa_2 R_{\mu\nu})\,\phi^\mu\phi^\nu]\,, \qquad (1)$$

where $\phi^\mu = \phi_\nu g^{\mu\nu}$. Einstein tensor $G_{\mu\nu}$ can be represented as $G_{\mu\nu} = \Theta_{\mu\nu}$, where the right hand side contains the third derivative terms

$$\Theta^{(3)}_{\mu\nu} = (\kappa_2 + 2\kappa_1)\,(g_{\mu\nu}\phi^\alpha\nabla_\alpha\Box\phi - \phi^\alpha\phi_{\alpha\mu\nu})\,, \qquad (2)$$

where ∇_λ is covariant derivative with respect to the Levi-Civita connection of $g_{\mu\nu}$,

$$\Box = \nabla_\lambda\nabla^\lambda, \quad \phi_{\mu\nu} = \nabla_\mu\phi_\nu, \quad \phi_{\alpha\mu\nu} = \nabla_\alpha\phi_{\mu\nu}$$

.

Similarly, the scalar equation

$$g^{\mu\nu}\phi_{\mu\nu} + \nabla_\mu\left[\phi_\nu(\kappa_1 g^{\mu\nu}R + \kappa_2 R^{\mu\nu})\right] = 0, \qquad (3)$$

in the general case contains the third derivatives of the metric. But for $-2\kappa_1 = \kappa_2$ the Ricci-terms are combined into the Einstein tensor satisfying $\nabla_\mu G^{\mu\nu} = 0$. Then the scalar equations becomes the second order $(g^{\mu\nu} + \kappa G^{\mu\nu})\,\phi_{\mu\nu} = 0$, while (2) disappears.

This theory, in the metric version, was found capable to provide inflationary mechanism without a potential Sushkov:2009, Unfortunately, now this metric model, apparently, is refuted by the speed of gravity waves measurement.

Palatini theory

The Palatini derivative scalar-tensor theory (hereinafter abbreviated PDST) has the same action $S = \int d^4x\sqrt{-g}\, L_P$, where in the Lagrangian

$$L = (\hat{R}_{\mu\nu} - \phi_\mu\phi_\nu)g^{\mu\nu} - \hat{R}_{\alpha\beta}\phi_\mu\phi_\nu(\kappa_1 g^{\alpha\beta}g^{\mu\nu} + \kappa_2 g^{\alpha\mu}g^{\beta\nu}) \quad (4)$$

the Ricci tensor $\hat{R}_{\mu\nu}$ is a function of the independent connection $\hat{\Gamma}^\lambda_{\mu\nu}$, while the Ricci scalar $\hat{R} = R_{\mu\nu}g^{\mu\nu}$ depends on the metric and on the connection. In the absence of fermions, when the Ricci tensor is contracted with symmetric tensors, the torsion can be consistently set equal to zero, so the variation of $\hat{R}_{\mu\nu}$ is

$$\delta\hat{R}_{\mu\nu} = \hat{\nabla}_\lambda\delta\hat{\Gamma}^\lambda_{\mu\nu} - \hat{\nabla}_\nu\delta\hat{\Gamma}^\lambda_{\mu\lambda},$$

where $\hat{\nabla}_\lambda \equiv \hat{\nabla}_\lambda(\hat{\Gamma})$ stands for covariant derivative with respect to the Palatini connection.

Integrating by parts, we arrive at the Euler-Lagrange equation:

$$\hat{\nabla}_\lambda \left(\sqrt{-g} W^{\mu\nu} \right) = 0, \quad W^{\mu\nu} = \lambda g^{\mu\nu} - \kappa_2 \phi^\mu \phi^\nu,$$

$$\lambda = (1 - \kappa_1 \psi), \quad \psi = \phi_\alpha \phi^\alpha.$$

Variation of the action with respect to the metric leads to the Einstein-Palatini equation

$$\lambda \hat{R}_{\mu\nu} - \phi_\mu \phi_\nu (1 + \kappa_1 \hat{R}) - 2\kappa_2 \hat{R}_{\alpha(\mu} \phi_{\nu)} \phi^\alpha - g_{\mu\nu} L/2 = 0. \qquad (5)$$

Finally, a variation over ϕ gives rise to a scalar equation

$$\partial_\mu \left[\sqrt{-g} \left(\phi^\mu + \kappa_1 \hat{R} \phi^\mu + \kappa_2 \hat{R}_{\alpha\beta} g^{\beta\mu} \phi^\alpha \right) \right] = 0,$$

which, in principle, can contain third derivatives of the metric and the fourth derivatives of the scalar field.

The standard way to find the connection $\hat{\Gamma}$ to transform the Palatini connection equation into an equation

$$\hat{\nabla}_\lambda \hat{g}_{\mu\nu} = 0$$

for some second metric $\hat{g}_{\mu\nu}$, in which case $\hat{\Gamma}$ can be identified with the Levi-Civita connection of the latter. For this, it is sufficient to ensure the following relation between the matrix $W^{\mu\nu}$ and the inverse metric $\hat{g}^{\mu\nu}$:

$$\sqrt{-g}W^{\mu\nu} = \sqrt{-\hat{g}}\hat{g}^{\mu\nu}, \quad \hat{g} = \det(\hat{g}_{\mu\nu}).$$

Then we get the equation in terms of the inverse new metric which is equivalent to what we are looking for.

The inverse matrix $W_{\mu\nu}$ reads:

$$W_{\mu\nu} = \lambda^{-1}\left(g_{\mu\nu} + \kappa_2 \Lambda^{-1}\phi_\mu\phi_\nu\right), \quad \Lambda = 1 - (\kappa_1 + \kappa_2)\psi.$$

Calculating the determimants one can find :

$$\hat{g} = g\,\Lambda\lambda^3.$$

Using this, the second metric can be represented as

$$\hat{g}_{\mu\nu} = \sqrt{\Lambda\lambda}\left(g_{\mu\nu} + \kappa_2 \Lambda^{-1}\phi_\mu\phi_\nu\right),$$

and the Palatini connection, will read:

$$\hat{\Gamma}^\lambda_{\mu\nu} = \frac{1}{2}\hat{g}^{\lambda\tau}\left(\partial_\mu \hat{g}_{\lambda\nu} + \partial_\mu \hat{g}_{\mu\lambda} - \partial_\lambda \hat{g}_{\mu\nu}\right).$$

Disformal transformation

The disformal transformation expresses the second metric $\hat{g}_{\mu\nu}$ through the physical metric $g_{\mu\nu}$, to which couples the matter. Not only the Palatini connection is expressed in terms of the dual metric $\hat{g}_{\mu\nu}$, but the full PDST theory will have a simpler form in the dual variables. Disformal dualities in non-minimal theories were extensively discussed recently (Bettoni:2013, Sakstein:2015, Domenech:2015, deRham:2016, Takahashi:2017).

The crucial question is whether the two theories related by non-pointlike transformations of variables are classically equivalent, with no extra degrees of freedom. A number of investigations Exirifard:2007,Deruelle:2014,Tsujikawa:2014,Domenech:2015,Arroja:2015, deRham:2016,Takahashi:2017) suggest that it is indeed the case, provided the disformal transformations are invertible.

Dual theory

Expressing the action in terms of the new metric we find for generic κ_1, κ_2 the Einstein-Hilbert gravity non-minimally coupled to ϕ:

$$S = \int \sqrt{-\hat{g}} \left[R_{\mu\nu}(\hat{g}) - \phi_\mu \phi_\nu \hat{\Lambda}^{-1} \right] \hat{g}^{\mu\nu} d^4 x,$$

where we denoted by $\hat{\Lambda}$ the scale factor Λ with $\psi = \phi_\mu \phi_\nu g^{\mu\nu}$ expressed through $\hat{\psi} = \phi_\mu \phi_\nu \hat{g}^{\mu\nu}$.
These two are related by the equation

$$\hat{\psi} = \psi(1 - \kappa_1 \psi)^{-3/2}([1 - (\kappa_1 + \kappa_2)\psi]^{1/2},$$

This relation becomes singular if one of the scale factors reaches zero, when the determinant ratio degenerates. We, therefore, demand $\lambda > 0, \Lambda > 0$ in the physical region of spacetime.

By the inverse function theorem, the solution $\psi(\hat{\psi})$ exists at any point where the derivative $d\hat{\psi}/d\psi \neq 0$. In our case this derivative is zero at $\psi = \psi_{\rm cr} = 2/(2\kappa_1 + 3\kappa_2)$. But there one easily finds

$$\hat{\psi}(\psi_{\rm cr}) = 2/(3\sqrt{3}\kappa_2).$$

To the right and to the left of this point, the function $\hat{\psi}(\psi)$ is monotonous (see Left panel in Fig.1). I general, the cubic equation has three solutions $\psi(\hat{\psi})$, from which one is real in the physical domain, and we naturally choose it. Thus, it is found that the disformal transformation is one-to-one and reversible indeed within the physical region. However, the existence of a physical region restricts the range of possible parameters (a more detailed discussion will be given elsewhere).

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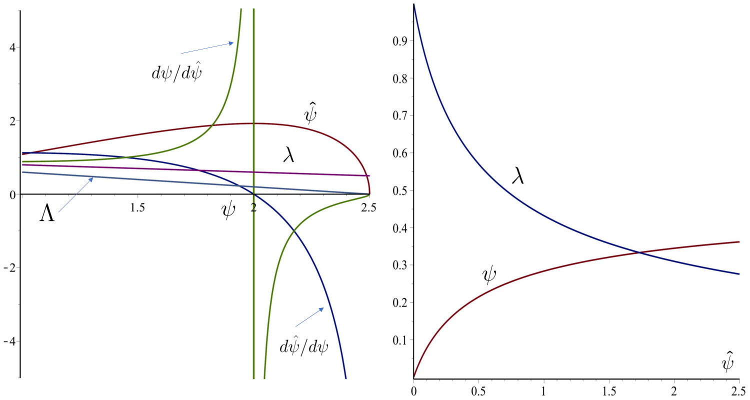

Рис.: Left panel: dependence $\hat{\psi}(\psi)$, derivatives $d\hat{\psi}/d\psi$, $d\psi/d\hat{\psi}$ and scale factors $\lambda(\psi)$, $\Lambda(\psi)$ for generic theory ($\kappa_1 = \kappa_2 = 0.2$. Right panel: solution $\psi(\hat{\psi})$ and scale factor λ for exceptional theory ($\kappa_1 = -\kappa_2 = 1$).

No ghosts

Note that we are still working in the Palatini formalism, so the Ricci tensor in be considered as a functional of the connection. But for such actions, both the metric and the Palatini formalism give the same equations, so, with some abuse on notation, we denote the Ricci tensor as metric one

Obviously, the Einstein-scalar theory does not suffer from Ostrogradski instabilities, so we conclude that the PDST theory with two generic coupling constants has no ghosts.

Exceptional theory

The case $\kappa_2 = -\kappa_1 \equiv \kappa$ is exceptional. Then $\Lambda = 1$, and the above theory reduces to the Einstein theory, minimally coupled to a massless scalar:

$$S = \int \sqrt{-\hat{g}}\left[R_{\mu\nu}(\hat{g}) - \phi_\mu\phi_\nu\right]\hat{g}^{\mu\nu}d^4x.$$

The Einstein equation reads

$$R_{\mu\nu} = \phi_\mu\phi_\nu,$$

and the scalar obeys the covariant d'Alembert equation

$$\hat{\Box}\phi = 0,$$

On shell the following conditions are satisfied: $L = 0$, $\hat{R} = \psi$, implying that the Palatini Einstein equation for $g_{\mu\nu}$ is valid.

Non-singular cosmology

Assume that $\kappa_2 = -\kappa_1 \equiv \kappa > 0$. Using disformal duality, we can construct an exact cosmological solutions of our theory starting with the spatially flat FRW cosmology in the Einstein's frame theory

$$d\hat{s}^2 = \hat{g}_{\mu\nu} dx^\mu dx^\nu = dt^2 - \hat{a}^2 \delta_{ij} dx^i dx^j.$$

The relevant Einstein and scalar field equations

$$R_{tt} = -\frac{3\ddot{\hat{a}}}{\hat{a}} = \dot{\phi}^2, \quad \ddot{\phi} + 3\frac{\dot{\hat{a}}}{\hat{a}}\dot{\phi} = 0$$

give a solution for "stiff-matter" cosmology proposed by Zel'dovich in 1972

$$\hat{a} = a_0 t^{1/3}, \quad \phi = \sqrt{2} \ln t / \sqrt{3}.$$

Obviously, it is singular at $t = 0$ and describes the decelerating expansion.

Now we derive a solution to our theory. Since the metric is diagonal and the scalar field depends only on t, we obtain a cubic algebraic equation for g_{tt}:

$$\left(|g_{tt}| - 2x/(3\sqrt{3})\right)^3 = |g_{tt}|, \quad x = \kappa\sqrt{3}/t^2.$$

Its solution is smooth, although in terms of real functions it looks piecewise:

$$|g_{tt}| = \frac{2x}{3\sqrt{3}} + \frac{1}{\sqrt{3}} \begin{cases} 2\cos\left(\frac{1}{3}\arccos(x)\right), & x < 1, \\ A^{1/3} + A^{-1/3}, & x > 1, \end{cases}$$

where $A = \left(x + \sqrt{x^2 - 1}\right)^{1/3}$. For large x (small t) one has:

$$|g_{tt}| = 2x/3\sqrt{3} + (2x)^{\frac{1}{3}}/\sqrt{3} + (4/x)^{\frac{1}{3}}/(2\sqrt{3}) + \ldots,$$

for small x (large t),

$$|g_{tt}| = 1 + x/\sqrt{3} - x^2/18 + \dots\,.$$

For g_{ij} one obtains :

$$g_{ij} = \delta_{ij} a^2, \quad a^2 = \hat{a}^2 |g_{tt}|^{1/3}.$$

Since $|g_{tt}| = 1$ only at large t, we need to go to the synchronous time $t \to \tau(t)$, so that $g_{\tau\tau} \equiv 1$, solving the equation

$$dt/d\tau = |g_{tt}|^{-1/2}.$$

For small t, keeping the leading term one finds:

$$dt/d\tau = H_0\, t \implies t = \mathrm{e}^{H_0 \tau},\ H_0 = \sqrt{3/(2\kappa)}.$$

For the spatial components for small t, one can find $a^2 \sim a_0^2 H_0^{-2/3}$, with no dependence on time at all. Finally, after scaling the spatial coordinates, we get the Minkowski metric for $t \to 0$. Theerefore, the space-time $g_{\mu\nu}$ starts at $\tau = -\infty$ as Minkowski space. Computing the second derivative of the scale factor $\ddot{a}$ one finds that the universe starts in accelerating phase which, however, ends soon with an insignificant gain of the cosmological radius (Fig. 2). After a short period of acceleration, the expansion decelerates and at large t the Zel'dovich's power law $a \sim t^{1/3}$ is restored, since the scale factor $\lambda \to 1$ (and the comoving time coincides with the Einstein frame comoving time).

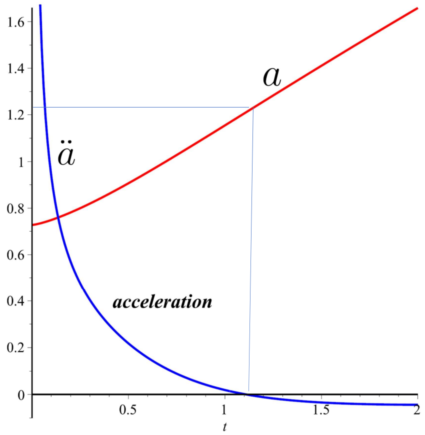

Рис.: Non-singular PDST cosmology. Expansion starts with finite scale factor with positive acceleration.

Although PDST metric is nonsingular at the initial moment of expansion, the scalar field diverges. In the Palatini theories it is assumed that matter couples to the metric $g_{\mu\nu}$, so geodesics, defined as curves of minimal length, do not stop at $t \to 0$, and the space is geodesically complete. At the same time, the auto-parallel curves, defined in terms of the Palatini connection, which coincides with the Levi-Civita connection of the singular metric will meet the singularity at $t = 0$. But, if matter couples to the metric and not to the connection, the scalar is not seen. This looks like Deus Ex Machina is coming in order to realize the cherished dream of General Relativity - removing of space-time singularities.

Static Geon

In the static case, interesting solutions arise for $\kappa_1 = -\kappa_2 > 0$, so here we denote $\kappa = \kappa_1$. Minimal scalar gravity has a well-known FJNW solution

$$\hat{g}_{tt} = -\hat{g}_{rr}^{-1} = -\left(1 - b/r\right)^{\gamma}, \; \hat{g}_{\theta\theta} = r^2 \left(1 - b/r\right)^{1-\gamma},$$

$$\phi = q(b)^{-1} \ln\left(1 - b/r\right),$$

where q is the scalar charge and $0 < \gamma < 1$, $\gamma = \left(1 - 4q^2/b^2\right)^{1/2}$. It is asymptotically flat and has a singular horizon at $r = b$. We want to find a PDST counterpart for this solution. The disformal transformation generates somewhat more complicated cubic equation for g_{rr}:

$$[g_{rr} - 2x/(3\sqrt{3})]^3 = w^2 g_{rr}, \quad w = \hat{g}_{rr} = (1 - b/r)^{-\gamma},$$

$$x = 3\sqrt{3}\kappa q^2/[2r^2(r-b)^2]$$

smooth real solution is:

$$g_{rr} = \frac{2x}{3\sqrt{3}} + \frac{1}{\sqrt{3}} \begin{cases} 2w \cos[\frac{1}{3} \arccos(x/w)], & x < w, \\ w^{2/3}B + w^{4/3}B^{-1}, & x > w, \end{cases}$$

where $B = \left(x + \sqrt{x^2 - w^2}\right)^{1/3}$. The remaining metric components then are:

$$g_{tt} = \hat{g}_{tt}/\lambda^{1/2},\ g_{\theta\theta} = \hat{g}_{\theta\theta}/\lambda^{1/2},\ \lambda = (g_{rr}/w)^{-2/3}.$$

At infinity $r \to \infty$, the variables $x \to 0,\ w \to 1$, while $g_{rr} \sim 1 + x/\sqrt{3}$, so $\lambda = 1 + O(r^{-4})$ and the solution remains asymptotically flat:

$$g_{tt} \sim -1 + \gamma b/r, \quad g_{rr} \sim 1 - \gamma b/r, \quad g_{\theta\theta} \sim r^2.$$

In the singularity $r = b$, one has $w \sim \xi^{-\gamma}, x \sim \xi^{-2}$, where $\xi = r/b - 1 \to 0$ leading to:

$$\lambda \sim \mu^{-2}\xi^{2(2-\gamma)/3}, \;\; g_{tt} \sim \mu\xi^{2(2\gamma-1)/3}, \;\; g_{rr} \sim \mu^3\xi^{-2},$$

$$g_{\theta\theta} \sim \mu b^2\xi^{(1-2\gamma)/3}, \;\; \mu = (\kappa q^2/b^4)^{1/3}$$

For $\gamma = 1/2$, the interval in the neighborhood of $r = b$ reads:

$$\mu^{-1}ds^2 = -dt^2 + (\mu dr/\xi)^2 + b^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

Passing to the new radial coordinate $\rho = b\ln\xi$ extending the domain $r \in (b, \infty)$ to a complete real line, one can find that the manifold is isomorphfic to the product $M_{1,1} \times S^2$ of the two-dimensional Minkowski space and a sphere of radius b. This manifold is geodesically complete. Therefore, our solution is a regular geon of the PDST theory. Its striking feature, however, that it is supported by a singular scalar.

PP-wave

Wave space-times (and more general classes of Kundt metrics) in Einstein theory coupled to the minimal scalar were constructed recently (Tahamtan:2015). Here we consider the simplest pp-wave spacetime

$$d\hat{s}^2 = F(u,x,y)\,du^2 - 2dudv + dx^2 + dy^2,$$

whose tensor Ricci is

$$R_{\mu\nu} = -\delta^u_\mu \delta^u_\nu \,\Delta\, F/2, \quad \Delta = \partial_x^2 + \partial_y^2.$$

Assuming that the scalar field depends only on u, it is found that the d'Alembert equation is satisfied: $\hat{\Box}\phi(u) = 0$, and the Einstein equation $R_{\mu\nu} = \phi_\mu \phi_\nu$ reduces to an equation for F:

$$\Delta\, F = -2\phi'^2.$$

Now construct the corresponding PDST solution. In this case, the disformal transformation is light-like . Assuming that the metric has nonzero only g_{uu}, g_{uv}, g_{ij}, we find only non-zero inverse metric components g^{vv}, g^{uv}, g^{ij}, therefore $\psi = \phi_\mu \phi_\nu g^{\mu\nu} = 0$, and the scale factor $\lambda = 1$. Then we easily find the pp-wave solution of the PDST theory:

$$d\hat{s}^2 = \left(F(u,x,y) - \kappa\phi'^2\right) du^2 - 2dudv + dx^2 + dy^2.$$

It is clear that it propagates at the speed of light.

Conclusion

1. PDST theory is related to the Einstein gravity, kinetically coupled to the scalar field, by means of a reversible disformal transformation and, consequently, it is free from Osrogradski ghosts.

2. In the case of opposite coupling constants $\kappa_1 = -\kappa_2$, the dual theory is simply an Einstein theory, minimally coupled to a massless scalar.

3. The duality transformation is exactly solvable. Using this, we have constructed cosmological, static spherically symmetric and pp-wave PDST solutions with intriguing properties: the cosmological solution has a completely non-singular metric, the static solution is geon-like

4 PDST pp-wave propagates exactly at the speed of light.

Thank You for Attention!