Homotopy Operators and Locality Theorems in Higher-Spin Equations

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OG, Vasiliev arXiv:1805.11941 [hep-th]

QFTG'2018, Tomsk, July 31

Introduction and Main Results

After the work of Giombi and Yin (2009) it is known that conventional homotopy (Vasiliev 1992)
can lead to nonlocalities beyond the free field level.
(Also Boulanger, Kessel, Skvortsov, Taronna 2015; Vasiliev 2017)

Aim of this talk is to show how to modify homotopy approach to reduce the degree of nonlocality in the sector of zero-forms in
all perturbation orders by introducing
Shifted Homotopies,
Z-dominance Lemma providing a sufficient condition controlling locality of dynamical field equations and
Pfaffian Locality Theorem showing how to choose shifted homotopies to achieve degeneracy of the Pfaffian matrix of derivatives over spinor variables in higher corrections in the zero-form sector.

For bilinear corrections PLT leads to local results of Vasiliev (2016,2017)
Didenko, O.G., Korybut and Vasiliev 2018 (talk of Slava Didenko)

Nonlinear HS Equations

$$\mathrm{d}_x\mathcal{W} + \mathcal{W} * \mathcal{W} = -i(dZ^A dZ_A + \eta\theta^\alpha\theta_\alpha B * k * \kappa + \bar\eta\bar\theta^{\dot\alpha}\bar\theta_{\dot\alpha} B * \bar k * \bar\kappa)$$

$$\mathrm{d}_x B + \mathcal{W} * B - B * \mathcal{W} = 0$$

Two-component spinor notations: $Y^A = (y^\alpha, \bar y^{\dot\alpha})$, $Z^A = (z^\alpha, \bar z^{\dot\alpha})$, $\alpha, \beta = 1, 2$

HS star product

$$(f * g)(Z;Y) = \int d^4U d^4V f(Z+U; Y+U)\, g(Z-V; Y+V)\, \mathrm{e}^{iU_A V^A},$$

$U_A V^A = u^\alpha v^\beta \epsilon_{\alpha\beta} + \bar u^{\dot\alpha}\bar v^{\dot\beta}\epsilon_{\dot\alpha\dot\beta}$, $sp(4)$**-invariant symplectic form** $(\epsilon_{\alpha\beta}, \epsilon_{\dot\alpha\dot\beta})$.

$\mathrm{d}_x = dx^m \frac{\partial}{\partial x^m}$ **space-time de Rham differential**: $\theta^\alpha = dz^\alpha$, $\bar\theta^{\dot\alpha} = d\bar z^{\dot\alpha}$,

κ **and** $\bar\kappa$ **-inner Klein operators**

$\kappa := \exp(iz_\alpha y^\alpha)$, $\kappa * \kappa = 1$, $\kappa * f(z^\alpha; y^\alpha; \theta^\alpha) = f(-z^\alpha; -y^\alpha; \theta^\alpha) * \kappa$

k **and** $\bar k$ **-outer Klein operators** $kf(z^\alpha; y^\alpha; \theta^\alpha) = f(-z^\alpha; -y^\alpha; -\theta^\alpha)\, k$

$\eta = \exp i\phi$ **is a free parameter**

Fields of the Nonlinear System

$$B(Z;Y;K|x)\,,\qquad \mathcal{W}(Z;Y;K|x) = (W_n(Z;Y;K|x)\mathrm{d}x^n\,,\; S_A(Z;Y;K|x)\theta^A)$$

Zero-forms $B(Z;Y;K|x)\;\;,\quad K=(k,\bar{k})\,,$

Spin- s physical fields: Z**-independent part** C **of** B

$$C_s(Y;K|x) = C_s^{1,0}(Y|x)k + C_s^{0,1}(Y|x)\bar{k}$$

$$C_s^{kj}(y,\bar{y}|x) = \frac{1}{2i}\sum_{|m-n|=2s}\frac{1}{m!n!}C^{kj}{}_{\alpha_1\ldots\alpha_n,\dot{\beta}_1\ldots\dot{\beta}_m}(x)y^{\alpha_1}\ldots y^{\alpha_n}\bar{y}^{\dot{\beta}_1}\ldots\bar{y}^{\dot{\beta}_m}$$

One-forms $W(Z;Y;K|x),\;\; S(Z;Y;K;|x)\,,$

Spin- s physical fields : Z**-independent part** ω **of** W,

$$\omega_s(Y;K|x) = \omega_s^{0,0}(Y|x) + \omega_s^{1,1}(Y|x)k\bar{k}$$

$$\omega_s(y,\bar{y};K|x) = \frac{1}{2i}\sum_{n+m=2(s-1)}\frac{1}{m!n!}\omega_{\alpha_1\ldots\alpha_n,\dot{\beta}_1\ldots\dot{\beta}_m}(k\bar{k}|x)y^{\alpha_1}\ldots y^{\alpha_n}\bar{y}^{\dot{\beta}_1}\ldots\bar{y}^{\dot{\beta}_m}$$

$C_1(y,\bar{y}|x)*C_2(y,\bar{y}|x)$ **contains arbitrary degrees of** $\partial_{1\alpha}\partial_2{}^{\alpha}\bar{\partial}_{1\dot{\alpha}}\bar{\partial}_2{}^{\dot{\alpha}}$

Source of nonlocality : $\frac{\partial}{\partial x^{\alpha\dot{\alpha}}} \sim \frac{\partial^2}{\partial y^{\alpha}\partial\bar{y}^{\dot{\alpha}}}$

Perturbative Analysis

Vacuum solution $B_0 = 0\,, \qquad S_0 = Z_A\theta^A\,, \qquad W_0 = \frac{1}{2}w^{AB}(x)Y_AY_B$

$$\mathsf{d}W_0 + W_0 * W_0 = 0$$

$$[S_0\,,f]_* = -2i\mathsf{d}_Z f\,, \qquad \mathsf{d}_Z = \theta^A\frac{\partial}{\partial Z^A}$$

$w^{AB}(x)$ **describes** AdS_4.

Generally: arbitrary flat background connection $W_0(Y)$

First-order: $B_1 = C(Y)\,, \quad S = S_0 + S_1\,,$

$$W = W_0(Y) + \omega(Y) + W_0(Y)C(Y)$$

Nontrivial space-time equations on $\omega(Y|x)$ **and** $C(Y|x)$**:**

d_Z**-cohomology**

Reconstruction of Z Variables: Conventional Homotopy

Perturbatively, order− n d_Z -dependent equations

$$\mathsf{d}_Z U_n(Z;Y|dZ) = V[U_{j<n}](Z;Y|dZ)\,, \qquad \mathsf{d}_Z V[U_{<n}](Z;Y|dZ) = 0$$

Solution

$$U_n(Z;Y|dZ) = \mathsf{d}^*_{\mathbf{Z}} V[U_{<n}](Z;Y|dZ) + h(Y) + \mathsf{d}_Z \epsilon(Z;Y|dZ)$$

Conventional homotopy operator $\partial = Z^A \frac{\partial}{\partial \theta^A}$ $\rightarrow$ **resolution d^*_Z**
(= contracting homotopy)

$$\mathsf{d}^*_{\mathbf{Z}} V(Z;Y|dZ) = Z^A \frac{\partial}{dZ^A} \int_0^1 \frac{dt}{t} V(tZ;Y|tdZ)$$

in some cases leads to nonlocalities beyond the free field level.

Shifted Homotopy

An obvious freedom in the definition of homotopy operator

$$Z^A \rightarrow Z^A + Q^A \,, \qquad \partial \rightarrow \left(Z^A + Q^A\right)\frac{\partial}{\partial\theta^A}\,, \qquad \frac{\partial}{\partial Z^A}(Q^B) = 0$$

$Q_A = \frac{\partial}{\partial Y^A}$ **acting on** $C(Y)$ **is admissible**

Resolution of identity is standard

$$\left\{\mathrm{d}_Z\,, \Delta_Q\right\} + \widehat{h}_Q = Id\,,$$

Resolution

$$\Delta_Q J\,(Z;Y;\theta) = \left(Z^A + Q^A\right)\frac{\partial}{\partial\theta^A}\int_0^1 d\tau\frac{1}{\tau}J\,(\tau Z - (1-\tau)Q;Y;\tau\theta)\,\,,$$

Cohomology projector

$$\widehat{h}_Q F(Z,Y) = F(-Q,Y)\,.$$

Conventional resolution $\mathrm{d}^*_Z = \Delta_0$

Exponential Representation and Z-Dominance Lemma

To control locality it suffices to consider the exponential parts of the operators acting on $C(y,\bar{y})$ focusing on the derivatives p^j $(\bar{p}^j)$.

Exponential representation in the holomorphic sector $\bar{\eta} = 0$

$$\underbrace{C(y) * \ldots * C(y)}_{n} = \exp i\Big(-\sum_j p^j{}_\alpha y^\alpha - \mathcal{P}^n(p)\Big) C(y_1)\ldots C(y_n)\Big|_{y_j=0}$$

$$p^j{}_\alpha := i\frac{\partial}{\partial y_j^\alpha}\,, \qquad \mathcal{P}^n(p) = \tfrac{1}{2}\textstyle\sum_{j,k} p^j{}_\alpha p^{k\alpha}$$

Solving $\mathrm{d}_Z U_n(Z;Y|dZ) = V[U_{j<n}](Z;Y|dZ)$ we use

Z-dominance Lemma: Since $Z-$dependence disappears on dynamical equations $\Rightarrow$ all terms dominated by the coefficients in front of the Z-dependent terms in the exponential factors in $U_n(Z;Y;K|x)$ trivialize on the field equations on the dynamical fields valued in the d_Z-cohomology.

$\mathbb{Z}_2$-Grading of Exponential Parts

The exponential parts in the holomorphic sector can be classified as:

Odd class $\mathcal{E}_n^1$:

$$\exp i(T(\tau)z_\gamma y^\gamma + A_j(\tau)p_\gamma^j z^\gamma + B_j(\tau)p_\gamma^j y^\gamma + \frac{1}{2}\mathcal{P}_{ij}(\tau)p_\gamma^i p^{j\gamma})k^{n+1}$$

$$\sum_{j=1}^{n}(-1)^j A_j = 0\,, \qquad \sum_{j=1}^{n}(-1)^j B_j = 1 - T\,, \qquad \sum_{i=1}^{n}(-1)^i \mathcal{P}_{ij} = -A_j$$

Even class $\mathcal{E}_n^0$:

$$\exp i(T(\tau)z_\gamma y^\gamma + A_j(\tau)p_\gamma^j z^\gamma + B_j(\tau)p_\gamma^j y^\gamma + \frac{1}{2}\mathcal{P}_{ij}(\tau)p_\gamma^i p^{j\gamma})k^{n}$$

$$\sum_{j=1}^{n}(-1)^j A_j = -T\,, \qquad \sum_{j=1}^{n}(-1)^j B_j = 0\,, \qquad \sum_{i=1}^{n}(-1)^i \mathcal{P}_{ij} = B_j.$$

Properties

- $$\mathcal{E}_n^j * \mathcal{E}_m^i \subset \mathcal{E}_{m+n}^{(j+i)|_2}$$

Shifted resolution with $Q = -iv^j p^j + \mu y$

$$\Delta_Q(\ldots)E_n(T,A,B,\mathcal{P}) = \int_0^1 d\tau(\ldots)E_n(\tau T, \tau A, B-(1-\tau)Tv-(1-\tau)\mu A, \mathcal{P}+\widetilde{\mathcal{P}}),$$

$$\widetilde{\mathcal{P}}_{ij} = (1-\tau)\left(A_j v_i - A_i v_j\right)$$

$$E_n(T,A,B,\mathcal{P},p|z,y) = \exp i(Tz_\gamma y^\gamma + A_j p_\gamma^j z^\gamma + B_j p_\gamma^j y^\gamma + \frac{1}{2}\mathcal{P}_{ij} p_\gamma^i p^{j\gamma})$$

generates mapping $\mathcal{M}_{Q,\tau} : E_n(T,A,B,\mathcal{P})k^m \to E_n(T',A',B',\mathcal{P}')k^m$

- $\sum_{j=1}^n (-1)^j v_j^1 = -1 \quad \Rightarrow \quad \mathcal{M}_{Q,\tau} : \mathcal{E}_n^1 \to \mathcal{E}_n^1 \quad \forall \tau, \mu$ **Odd resolutions**
- $\sum_{j=1}^n (-1)^j v_j^0 = \mu \quad \Rightarrow \quad \mathcal{M}_{Q,\tau} : \mathcal{E}_n^0 \to \mathcal{E}_n^0 \quad \forall \tau, \mu$ **Even resolutions**

$\mu = -1$ **: the two conditions are equivalent**

Differential operator d_x:

$$\mathsf{d}_x C(Y_1)\dots C(Y_n) = \sum_i C(Y_1)\dots C\widehat{(Y_i)}\dots C(Y_n)\Big|_{C(Y_i)\to\sum_m J_m(Y_i)}$$

by virtue of field equations

$$\mathsf{d}_x C(Y;K|x) = \sum_m J_m(\underbrace{C(Y),\dots,C(Y)}_{m})$$

defines mapping

- $$S_{i,E^1_m(0,0,B,P,p|0,y)} : \mathcal{E}^p_n \to \mathcal{E}^p_{n+m-1}\,, \qquad p = 0,1$$

Structure Lemma in the Holomorphic Sector

Corrections to dynamical equations to any perturbation order
are constructed inductively, starting from

$$\gamma = \exp\left(iz_\alpha y^\alpha\right) k\theta^2 \quad \& \quad C(y,\bar{y}|x) \quad \& \quad W_0(y,\bar{y}|x) \quad \& \quad \omega(y,\bar{y}|x)$$

applying star-product, shift resolutions and d_x-differentiation.

If shifted resolutions $\triangle$ in the one-form sector are even
while those in the zero-form sector are odd up to some order
then, inductively,

B_j are in the odd class $\mathcal{E}^1$, $\mathcal{W}_j$ are in the even class $\mathcal{E}^0$

in this order as well

Pfaffian Locality in the Holomorphic Sector

Odd fields contain Pfaffian matrix $\mathcal{P}_{ij}$ satisfying $\sum_{i=1}^{n}(-1)^i\mathcal{P}_{ij}=-A_j$.

Coefficients A_j are in front of the $z^\alpha p^j{}_\alpha$

$\Rightarrow$ A_j **trivialize in the field equations** $\Rightarrow \sum_{i=1}^{n}(-1)^i\mathcal{P}_{ij}=0$

$\Rightarrow$ $\mathcal{P}_{ij}$ **is degenerate leading to the smaller degree of nonlocality**

than $C_1(y)*\ldots*C_n(y)$ **at least for even** n.

In the second order Pfaffian locality implies usual locality

Conclusion

The class of shifted homotopy operators is introduced

Z-Dominance Lemma and Structure Lemma are proven

leading to

Pfaffian Locality Theorem describing a class of shifted resolutions

decreasing the level of nonlocality in higher orders that leads to

the local result in the first nontrivial order