Two-spinor description of massive particles and relativistic spin projection operators.

A. P. Isaev [1,2,3],
in collaboration with M. A. Podoinitsyn [1,2]

[1] ***Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna***

[2] ***State University of Dubna, Dubna, Russia***

[3] ***Lomonosov Moscow State University, Moscow, Russia***

Nucl.Phys. B929 (2018) 452-484

QFT & G ⋄ 2018,
July 30 - August 4, 2018, TSPU, Tomsk, Russia

Plan

1. **Introduction.**
2. **Polarization vectors for spin-tensor fields**
   - Massive unitary representations of the group $ISL(2,\mathbb{C})$.
   - Spin-tensor reps of $ISL(2,\mathbb{C})$ and Dirac-Pauli-Fierz equations
   - The polarization vectors for fields of arbitrary integer spin
3. **Spin projection operators for integer and half-integer spins**

Introduction.

In this report, using the Wigner unitary irreps of the covering group $ISL(2, C)$, which covers the Poincaré group $ISO^{\uparrow}(1, 3)$, we construct spin-tensor wave functions of a special kind. These spin-tensor wave functions form spaces of irreducible representations of the group $ISL(2, C)$ and automatically satisfy the Dirac-Pauli-Fierz wave equations for free massive particles of arbitrary spin. We use the approach set forth in S. Weinberg, Phys. Rev. **133** (1964) B1318; **134** (1964) B882; see also books: 1) *"Introduction to Elementary Particle Theory"* by Yu. Novozhilov, 2) *"Ideas and Methods of Supersymmetry and Supergravity: Or a Walk Through Superspace"* by I. Buchbinder and S. Kuzenko, 3) *"Theory of Groups and Symmetries. Part II"* (in preparation) by A. Isaev and V. Rubakov, ... The construction is carried out with the help of Wigner operators which translate unitary massive representation of the group $ISL(2, C)$ (induced from the irreps of the little subgroup $SU(2)$) acting in the space of Wigner wave functions to representations of the group $ISL(2, C)$, acting in the space of special massive spin-tensor fields.

Here a special parametrization of Wigner operators is proposed, with the help of which the momenta of particles on the mass shell and solutions of the Dirac-Pauli-Fierz wave equations are rewritten in terms of a pair of Weyl spinors (*two-spinor formalism* K.P. Tod, L.P. Hughston, S. Fedoruk, J. Lukierski, J. A. de Azcarraga and many others). The expansion of a completely symmetric Wigner wave function over a specially chosen basis provides a natural recipe for describing polarizations of massive particles with arbitrary spins. As the application of this formalism, a generalization of the Behrends-Fronsdal projection operator is constructed, which determines the spin-tensor structures of the two-point Green function (propagator) of massive particles with any higher spins in the case of arbitrary space-time dimension D. Note that these spin projection operators are also employed for analysis of the high energy scattering amplitudes, differential cross sections, etc.

Massive unitary representations of $ISL(2,\mathbb{C})$

To fix the notation, we recall the definition of the covering group $ISL(2,\mathbb{C})$ of the Poincare group $ISO^{\uparrow}(1,3)$. The group $ISL(2,\mathbb{C})$ is the set of all pairs (A,X), where $A \in SL(2,\mathbb{C})$, and X is any Hermitian (2×2) matrix which can always be represented in the form ($x_m \in \mathbb{R}$)

$$X = x_0\,\sigma^0 + x_1\,\sigma^1 + x_2\,\sigma^2 + x_3\,\sigma^3 = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \in \mathbf{H}\,.$$

The multiplication in the group $ISL(2,\mathbb{C})$ is given by the formula

$$(A',Y')\cdot(A,Y) = (A'\cdot A,\; A'\cdot Y\cdot A'^{\dagger} + Y')\,.$$

The $ISL(2,\mathbb{C})$ group action in the Minkowski space $\mathbb{R}^{1,3} = \mathbf{H}$

$$(A,Y)\cdot X = A\cdot X\cdot A^{\dagger} + Y \;\in\; \mathbf{H}\,, \qquad \forall\, X,Y \in \mathbf{H}\,. \tag{1}$$

The action of the $SL(2,\mathbb{C})$ group on vectors x in the Minkowski space $\mathbb{R}^{1,3}$ is:

$$X \;\to\; X' = A \cdot X \cdot A^{\dagger} \;\;\Rightarrow$$

$$\sigma^k x'_k = \sigma^k \, \Lambda_k{}^m(A)\, x_m \;\;\Rightarrow\;\; x'_k = \Lambda_k{}^m(A)\, x_m \;,$$

where $X_{\alpha\dot{\beta}} = x_k \; \sigma^k_{\alpha\dot{\beta}}$, $(\alpha, \dot{\beta} = 1, 2)$ and the (4×4) matrix $||\Lambda^m{}_k(A)|| \in SO^{\uparrow}(1,3)$ is determined from the relations

$$\underline{A \cdot \sigma^m \cdot A^{\dagger} = \sigma^k \, \Lambda_k{}^m(A)} \;\Leftrightarrow\; A_{\xi}{}^{\alpha}\, A^{*}_{\dot{\gamma}}{}^{\dot{\beta}}\, \sigma^m_{\alpha\dot{\beta}} = \sigma^k_{\xi\dot{\gamma}}\, \Lambda_k{}^m(A)\;.$$

We need also to have dual set of σ-matrices:

$$\tilde{\sigma}^m = (\tilde{\sigma}^0, -\tilde{\sigma}^1, -\tilde{\sigma}^2, -\tilde{\sigma}^3)\;, \quad (\tilde{\sigma}^m)^{\dot{\alpha}\beta}\;.$$

Further we will consider mostly the massive case: $\mathrm{m} > 0$. In this case the unitary irreps of the group $ISL(2,\mathbb{C})$ are characterized by spin $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$ and act in the spaces of Wigner wave functions $\phi_{(\alpha_1 \dots \alpha_{2j})}(k)$, which are components of a completely symmetric $SU(2)$-tensor of rank $2j$. Here $k = (k_0, k_1, k_2, k_3)$ denotes the four-momentum of a particle with mass m:

$$(k)^2 = k^n k_n = k_\ell \eta^{\ell n} k_n = k_0^2 - k_1^2 - k_2^2 - k_3^2 = \mathrm{m}^2 \; .$$

Let us fix some test momentum $q = (q_0, q_1, q_2, q_3)$ such that $(q)^2 = \mathrm{m}^2$, $q_0 > 0$ and choose a representative $A_{(k)} \in SL(2,\mathbb{C})$:

$$\boxed{(k\sigma) = A_{(k)} \cdot (q\sigma) \cdot A^\dagger_{(k)}} \;\Leftrightarrow\; \quad k_m = (\Lambda_k)_m^{\;\;n} \, q_n \; ,$$

where $(k\sigma) = k^n \sigma_n$, $(q\sigma) = q^n \sigma_n$. The relation between the matrices $A_{(k)}$ and $\Lambda_k \equiv \Lambda(A_{(k)})$ is standard.

Define a *stability subgroup* (*little group*) $G_q \subset SL(2,\mathbb{C})$ of the momentum q as the set of matrices $A \in SL(2,\mathbb{C})$ satisfying the condition

$$A \cdot (q\sigma) \cdot A^{\dagger} = (q\sigma) \;\;\Leftrightarrow\;\; A_{\alpha}{}^{\gamma}\,(q^n\sigma_n)_{\gamma\dot{\alpha}}\,(A^{*})_{\dot{\gamma}}{}^{\dot{\alpha}} = (q^n\sigma_n)_{\alpha\dot{\gamma}}\;.$$

In the massive case $(q)^2 = \mathrm{m}^2$, $\mathrm{m} > 0$, the stability subgroup G_q is isomorphic to $SU(2)$ for any choice of test momenta q. The matrix $A_{(k)} \in SL(2,\mathbb{C})$ is defined up to right multiplication $A_{(k)} \to A_{(k)} \cdot U$ by an element U of the stability subgroup $G_q = SU(2)$:

$$(A_{(k)} \cdot U) \cdot (q\sigma) \cdot (A_{(k)} \cdot U)^{\dagger} = A_{(k)} \cdot (U \cdot (q\sigma) \cdot U^{\dagger}) \cdot A^{\dagger}_{(k)} = (k\sigma)\;.$$

For each k we fix a unique matrix $A_{(k)}$ which numerate left coset in $SL(2,\mathbb{C})$ with respect to the subgroup $G_q = SU(2)$, i.e. matrices $A_{(k)}$ are points in the coset space $SL(2,\mathbb{C})/SU(2)$.

Explicit formula for unitary irreps of $ISL(2,\mathbb{C})$

Let $T^{(j)}$ be a finite-dimensional irreducible $SU(2)$ representation with spin j, acting in the space of symmetric spin-tensors $\phi_{(\alpha_1 \dots \alpha_{2j})}$. The Wigner unitary irreducible representations $\mathcal{U}$ of the group $ISL(2,\mathbb{C})$ with spin j are defined by the following action of the element $(A,a) \in ISL(2,\mathbb{C})$ in the space of wave functions (WFs) $\phi_{(\alpha_1,\dots,\alpha_{2j})}(k)$:

$$[\mathcal{U}(A,a)\cdot\phi]_{\bar{\alpha}}(k) \equiv \phi'_{\bar{\alpha}}(k) = e^{ia^m k_m}\; T^{(j)}_{\bar{\alpha}\bar{\beta}}(h_{A,\Lambda^{-1}\cdot k})\; \phi_{\bar{\beta}}(\Lambda^{-1}\cdot k)\;.$$

Here we use the concise notation $\phi_{\bar{\alpha}}(k) \equiv \phi_{(\alpha_1 \dots \alpha_{2j})}(k)$, the indices $\bar{\alpha}, \bar{\beta}$ are multi-indices $(\alpha_1 \dots \alpha_{2j})$, $(\beta_1 \dots \beta_{2j})$, the matrix $\Lambda \in SO^{\uparrow}(1,3)$ is related to $A \in SL(2,\mathbb{C})$ in standard way, and the element (dependent on k)

$$h_{A,\Lambda^{-1}\cdot k} = A^{-1}_{(k)} \cdot A \cdot A_{(\Lambda^{-1}\cdot k)} \;\;\in\;\; SU(2)\;,$$

belongs to the stability subgroup $SU(2) \subset SL(2,\mathbb{C})$.

In the formula for Wigner representation $\mathcal{U}$ the element $h_{A,\Lambda^{-1}\cdot k}$ of the stability subgroup is taken in the representation $T^{(j)}$ as matrix $||T^{(j)}_{\bar{\alpha}\bar{\alpha}'}(h_{A,\Lambda^{-1}\cdot k})||$ which can be written in the factorized form

$$T^{(j)}_{\bar{\beta}\bar{\alpha}}(h_{A,\Lambda^{-1}\cdot k}) = \left[\left(h_{A,\Lambda^{-1}\cdot k}\right)_{\beta_1}^{\ \alpha_1} \cdots \left(h_{A,\Lambda^{-1}\cdot k}\right)_{\beta_{p+r}}^{\ \alpha_{p+r}} \right] =$$
$$= \left[\left(h_{A,\Lambda^{-1}\cdot k}\right)_{\beta_1}^{\ \alpha_1} \cdots \left(h_{A,\Lambda^{-1}\cdot k}\right)_{\beta_p}^{\ \alpha_p} \right] \cdot$$
$$\cdot \left[\left((q\tilde{\sigma})^{-1} \cdot h^{\dagger\,-1}_{A,\Lambda^{-1}\cdot k} \cdot (q\tilde{\sigma})\right)_{\beta_{p+1}}^{\ \alpha_{p+1}} \cdots \left((q\tilde{\sigma})^{-1} \cdot h^{\dagger\,-1}_{A,\Lambda^{-1}\cdot k} \cdot (q\tilde{\sigma})\right)_{\beta_{p+r}}^{\ \alpha_{p+r}} \right] .$$

Here we split the tensor product of $2j = (p + r)$ factors $h_{A,\Lambda^{-1}\cdot k}$ into two groups. The first group consists of the p factors $h_{A,\Lambda^{-1}\cdot k}$, and in the second group we write r multipliers $h_{A,\Lambda^{-1}\cdot k}$ as $\left((q\tilde{\sigma})^{-1} \cdot h^{\dagger\,-1}_{A,\Lambda^{-1}\cdot k} \cdot (q\tilde{\sigma})\right)$ by using the generalized unitarity condition

$$h = (q\tilde{\sigma})^{-1} \cdot (h^{-1})^{\dagger} \cdot (q\tilde{\sigma}) \quad \Leftrightarrow \quad h^{\dagger} \cdot (q\tilde{\sigma}) \cdot h = (q\tilde{\sigma}) ,$$

for the elements h of little group $G_q = SU(2)$.

Now we use factorized form of the matrix $h_{A,\Lambda^{-1}\cdot k} = A^{-1}_{(k)} \cdot A \cdot A_{(\Lambda^{-1}\cdot k)}$, substitute it into the factorized matrix $T^{(j)}(h_{A,\Lambda^{-1}\cdot k}) =$

$$= [A^{\otimes p}_{(k)} \otimes (A^{\dagger -1}_{(k)}(q\tilde{\sigma}))^{\otimes r}]^{-1} \cdot [A^{\otimes p} \otimes (A^{\dagger -1})^{\otimes r}] \cdot [A^{\otimes p}_{(\Lambda^{-1}\cdot k)} \otimes (A^{\dagger -1}_{(\Lambda^{-1}\cdot k)}(q\tilde{\sigma}))^{\otimes r}]$$

introduce (instead of the Wigner WFs $\phi_{(\delta_1 \ldots \delta_{p+r})}(k)$) spin-tensor fields of $(\frac{p}{2}, \frac{r}{2})$-type (with r dotted and p undotted indices):

$$\psi^{(r)(\dot{\beta}_1 \ldots \dot{\beta}_r)}_{(\alpha_1 \ldots \alpha_p)}(k) = \mathrm{m}^{-r} \left[[A^{\otimes p}_{(k)} \otimes (A^{\dagger -1}_{(k)}(q\tilde{\sigma}))^{\otimes r}] \cdot \phi(k) \right]^{(\dot{\beta}_1 \ldots \dot{\beta}_r)}_{(\alpha_1 \ldots \alpha_p)} =$$

$$= \frac{1}{\mathrm{m}^r} (A_{(k)})^{\delta_1 \ldots \delta_p}_{\alpha_1 \ldots \alpha_p} \cdot (A^{-1\dagger}_{(k)} \cdot (q\tilde{\sigma}))^{\dot{\beta}_{p+1} \ldots \dot{\beta}_{p+r}; \delta_{p+1} \ldots \delta_{p+r}} \phi_{(\delta_1 \ldots \delta_p \delta_{p+1} \ldots \delta_{p+r})}(k) \, .$$

and rewrite the $ISL(2,\mathbb{C})$-representation $\mathcal{U}$ as following

$$[\mathcal{U}(A,a) \cdot \psi^{(r)}]^{(\dot{\beta}_1 \ldots \dot{\beta}_r)}_{(\alpha_1 \ldots \alpha_p)}(k) =$$

$$= e^{ia^m k_m} \left[A^{\gamma_1 \ldots \gamma_p}_{\alpha_1 \ldots \alpha_p} \; (A^{\dagger -1})^{\dot{\beta}_1 \ldots \dot{\beta}_r}_{\dot{\kappa}_1 \ldots \dot{\kappa}_r} \right] \; \psi^{(r)(\dot{\kappa}_1 \ldots \dot{\kappa}_r)}_{(\gamma_1 \ldots \gamma_p)}(\Lambda^{-1} \cdot k) \, ,$$

where $A^{\cdots}_{\cdots} \, (A^{\dagger -1})^{\cdots}_{\cdots} = [A^{\otimes p} \otimes (A^{\dagger -1})^{\otimes r}]^{\cdots}_{\cdots}$, $A \in SL(2,\mathbb{C})$.
The upper index (r) of the spin-tensors $\psi^{(r)}$ distinguishes these spin-tensors with respect to the number of dotted indices.

The operators $A_{(k)}^{\otimes p} \otimes \left(\frac{1}{\mathrm{m}} A_{(k)}^{\dagger -1}(q\tilde{\sigma})\right)^{\otimes r}$ which convert Wigner wave functions $\phi(k)$ into spin-tensor fields $\psi^{(r)}(k)$ of $(\frac{p}{2}, \frac{r}{2})$-type are called *the Wigner operators*.

Proposition 1. The wave functions $\psi^{(r)}$ satisfy the *Dirac-Pauli-Fierz (DPF) equations* [P.A.M.Dirac (1936), M. Fierz and W. Pauli (1939)]:

$$k^m (\tilde{\sigma}_m)^{\dot{\gamma}_1 \alpha_1} \psi^{(r)(\dot{\beta}_1 \dots \dot{\beta}_r)}_{(\alpha_1 \dots \alpha_p)}(k) = \mathrm{m}\, \psi^{(r+1)(\dot{\gamma}_1 \dot{\beta}_1 \dots \dot{\beta}_r)}_{(\alpha_2 \dots \alpha_p)}(k) \, , \quad (r = 0, \dots, 2j-1) \, ,$$

$$k^m (\sigma_m)_{\gamma_1 \dot{\beta}_1} \psi^{(r)(\dot{\beta}_1 \dots \dot{\beta}_r)}_{(\alpha_1 \dots \alpha_p)}(k) = \mathrm{m}\, \psi^{(r-1)(\dot{\beta}_2 \dots \dot{\beta}_r)}_{(\gamma_1 \alpha_1 \dots \alpha_p)}(k) \, , \quad (r = 1, \dots, 2j) \, ,$$

which describe the dynamics of a massive relativistic particle with spin $j = (p + r)/2$. The compatibility conditions for the system of DPF equations are given by the mass shell relations $(k^n k_n - m^2)\, \psi^{(r)}(k) = 0$.

Proof. Use the definitions of matrices $A_{(k)} \in SL(2, \mathbb{C})/SU(2)$:

$$(k\tilde{\sigma}) \cdot A_{(k)} = A_{(k)}^{\dagger -1} \cdot (q\tilde{\sigma}) \, , \quad (k\sigma) \cdot A_{(k)}^{\dagger -1} = A_{(k)} \cdot (q\sigma) \, .$$

In the case of $p+r=2j$, the system of spin-tensor wave functions $\psi^{(r)}$ which obey the Dirac-Pauli-Fierz equations describes relativistic particles with spin j.

Proposition 2. Spin-tensor wave functions $\psi^{(r)(\dot\beta_1\ldots\dot\beta_r)}_{(\alpha_1\ldots\alpha_p)}(k)$ of type $(\frac{p}{2},\frac{r}{2})$, which obey the Dirac-Pauli-Fierz equations, automatically satisfy the equations

$$[(\hat W^m\,\hat W_m)\,\psi]^{(r)(\dot\beta_1\ldots\dot\beta_r)}_{(\alpha_1\ldots\alpha_p)}(k) = -\mathrm{m}^2 j(j+1)\,\psi^{(r)(\dot\beta_1\ldots\dot\beta_r)}_{(\alpha_1\ldots\alpha_p)}(k)\ ,$$

where $j=(\frac{p}{2}+\frac{r}{2})$, $\hat W_m$ are the components of the Pauli-Lubanski vector

$$\hat W_m = \frac{1}{2}\varepsilon_{mnij}M^{ij}P^n = \frac{1}{2}\varepsilon_{mnij}\hat\Sigma^{ij}P^n\ ,$$

and $\hat W_m\hat W^m$ is the Casimir operator for the group $ISL(2,\mathbb{C})$; $\hat\Sigma^{ij}$ — spin part of M^{ij}.

The matrices $A_{(k)}$ numerate points of the coset space $SL(2,\mathbb{C})/SU(2)$. The left action of the group $SL(2,\mathbb{C})$ on $SL(2,\mathbb{C})/SU(2)$ is

$$A \cdot A_{(k)} = A_{(\Lambda \cdot k)} \cdot U_{A,k} \; , \qquad A \in SL(2,\mathbb{C}) \; , \;\; \Lambda \in SO^{\uparrow}(1,3) \; ,$$

where matrices A and Λ are related by standard formula $A\vec{\sigma}A^{\dagger} = \Lambda\vec{\sigma}$ and the element $U_{A,k} \in SU(2)$ depends on A and momentum k. Under this left action the element $A \in SL(2,\mathbb{C})$ transforms two columns of the matrix $A_{(k)}$ as two Weyl spinors. Therefore, it is convenient to represent the matrix $A_{(k)}$ by using two Weyl spinors μ and λ with components $\mu_\alpha, \; \lambda_\alpha$ (the matrix $A^{\dagger}_{(k)}$ will be correspondingly expressed in terms of the conjugate spinors $\overline{\mu}$ and $\overline{\lambda}$) in the following way:

$$(A_{(k)})_{\alpha}{}^{\beta} = \frac{1}{(\mu^{\rho}\,\lambda_{\rho})^{1/2}} \begin{pmatrix} \mu_1 & \lambda_1 \\ \mu_2 & \lambda_2 \end{pmatrix} \;\Rightarrow\; (A^{\dagger -1}_{(k)})^{\dot{\alpha}}{}_{\dot{\beta}} = \frac{1}{(\overline{\mu}^{\dot{\rho}}\,\overline{\lambda}_{\dot{\rho}})^{1/2}} \begin{pmatrix} \overline{\lambda}_{\dot{2}} & -\overline{\mu}_{\dot{2}} \\ -\overline{\lambda}_{\dot{1}} & \overline{\mu}_{\dot{1}} \end{pmatrix} ,$$

In the case $q = (\mathrm{m}, 0, 0, 0)$, it follows from

$$(k\sigma) = A_{(k)}\,(q\sigma)\,A^{\dagger}_{(k)}\ ,$$

that the momentum k is expressed in terms of the spinors $\mu_\alpha, \lambda_\beta, \overline{\mu}_{\dot\beta}, \overline{\mu}^{\dot\beta}$ as follows:

$$\frac{\mathrm{m}}{|\mu^\alpha\lambda_\alpha|}(\mu_\alpha\overline{\mu}_{\dot\beta} + \lambda_\alpha\overline{\lambda}_{\dot\beta}) = (k^n\sigma_n)_{\alpha\dot\beta}\ ,\quad \frac{\mathrm{m}}{|\mu^\alpha\lambda_\alpha|}(\mu^\alpha\overline{\mu}^{\dot\beta} + \lambda^\alpha\overline{\lambda}^{\dot\beta}) = (k^n\widetilde{\sigma}_n)^{\dot\beta\alpha}\ .$$

These two-spinor expressions for the four-vector k ($k^2 = \mathrm{m}^2$ and $k_0 > 0$) are generalizations of the well-known Penrose twistor representation for momentum k of a massless particle. The wave functions of massive relativistic particles, which are functions of a four-momentum k, can be considered as functions of two Weyl spinors λ and μ.

One can show that the two-spinor description of massive particles based on the representation described above proves to be extremely convenient in describing polarization properties of massive particles with arbitrary spin j.

The expansion of the spin-tensor fields over the polarization vectors of arbitrary integer spin is

$$\psi^{(r)(\dot\beta_1\dots\dot\beta_r)}_{(\alpha_1\dots\alpha_p)}(k) = \frac{1}{\sqrt{(2j)!}} \sum_{m=-j}^{j} \phi_m(k) \overset{(m)}{e}{}^{(\dot\beta_1\dots\dot\beta_r)}_{(\alpha_1\dots\alpha_p)}(k) \, ,$$

where the polarization vectors are

$$\overset{(m)}{e}{}^{(\dot\beta_1\dots\dot\beta_r)}_{(\alpha_1\dots\alpha_p)}(k) = \frac{1}{\sqrt{(2j)!}} \prod_{i=1}^{p} (A_{(k)})_{\alpha_i}{}^{\rho_i} \prod_{\ell=1}^{r} \left(A^{-1\dagger}_{(k)} \, \tilde\sigma_0\right)^{\dot\beta_\ell \rho_{p+\ell}} \epsilon^{(m)}_{\rho_1\cdots\rho_{2j}} ,$$

The constant symmetric tensor $\epsilon^{(m)}_{\rho_1\cdots\rho_{2j}}$ is defined as

$$\epsilon^{(m)}_{\underbrace{1\cdots1}_{j+m}\underbrace{2\cdots2}_{j-m}} = \sqrt{(j+m)!(j-m)!} \, ,$$

and for other components we have $\epsilon^{(m)}_{\rho_1\cdots\rho_{2j}} = 0$. If we express matrices $A_{(k)}$ in terms of spinors λ and μ we obtain explicit formulas for polarization vector-spinors in terms of λ and μ.

Proposition 3. The spin-tensors $\overset{(m)}{e}$ for $r = p = j$ (integer spins): $\overset{(m)}{e}{}^{(\dot\beta_1 \dots \dot\beta_j)}_{(\alpha_1 \dots \alpha_j)}$ satisfy the relations:

$$\overset{(m+1)}{e}{}^{(\dot\beta_1 \dots \dot\beta_j)}_{(\alpha_1 \dots \alpha_j)} = \frac{1}{\sqrt{(j-m)(j+m+1)}} \left(\mu_\gamma \frac{\partial}{\partial \lambda_\gamma} - \overline{\lambda}^{\dot\gamma} \frac{\partial}{\partial \overline{\mu}^{\dot\gamma}} \right) \overset{(m)}{e}{}^{(\dot\beta_1 \dots \dot\beta_j)}_{(\alpha_1 \dots \alpha_j)}$$

where μ_γ, λ_γ, $\overline{\mu}^{\dot\gamma}$, $\overline{\lambda}^{\dot\gamma}$ - Weyl spinors.

Remark. In the case $r = p$ (i.e. for integer spins j) we deduce the recurrence relation:

$$e^{(m)}_{n_1 \cdots n_j} = \frac{1}{\sqrt{2j\,(2j-1)}} \Big(\sqrt{(j+m)(j+m-1)}\, e^{(m-1)}_{n_1 \cdots n_{j-1}}\, e^{(+)}_{n_j} +$$

$$+ \sqrt{(j-m)(j-m-1)}\, e^{(m+1)}_{n_1 \cdots n_{j-1}}\, e^{(-)}_{n_j} + \sqrt{2(j-m)(j+m)}\, e^{(m)}_{n_1 \cdots n_{j-1}}\, e^{(0)}_{n_j} \Big) \; ,$$

which completely determines the polarization tensor $e^{(m)}_{n_1 \cdots n_j}$ for any j via the vectors of polarization $e^{(a)}_{n}$ $(a = 0, \pm)$ for $j = 1$.

Spin projection operator $\Theta(k)$.

First we consider the case of integer spins j.
We construct the spin projection operator $\Theta(k)$ as the sum of products $\mathrm{e}^{(m)}(k) \cdot \overline{\mathrm{e}}^{(m)}(k)$ over all polarizations m:

$$\Theta^{n_1 \cdots n_j}_{r_1 \cdots r_j}(k) := (-1)^j \sum_{m=-j}^{j} \mathrm{e}^{(m)}_{r_1 \cdots r_j}(k) \overline{\mathrm{e}}^{(m) n_1 \cdots n_j}(k)$$

This operator is sometimes called the density matrix for a massive particle with integer spin j, or the Behrends-Fronsdal projection operator (R.E. Behrends, C. Fronsdal (1957)).

For spin $j = 1$ the operator $\Theta(k)$ is well known

$$\Theta^{(1)}_{nm}(k) = \left(\eta_{nm} - \frac{k_n k_m}{\mathrm{m}^2}\right) = \left(\eta_{nm} - \frac{k_n k_m}{k^2}\right) .$$

Proposition 4. The operator $\Theta(k)$, defined as $\sum_m \mathrm{e}^{(m)}(k)\bar{\mathrm{e}}^{(m)}(k)$, satisfies the following properties:

1) projective property and reality: $\Theta^2 = \Theta, \;\; \Theta^\dagger = \Theta$;

2) symmetry: $\Theta^{n_1 \cdots n_j}_{\cdots r_i \cdots r_\ell \cdots} = \Theta^{n_1 \cdots n_j}_{\cdots r_\ell \cdots r_i \cdots}, \;\; \Theta^{\cdots n_i \cdots n_\ell \cdots}_{r_1 \cdots r_j} = \Theta^{\cdots n_\ell \cdots n_i \cdots}_{r_1 \cdots r_j}$;

3) transversality: $k^{r_1}\Theta^{n_1 \cdots n_j}_{r_1 \cdots r_j} = 0, \;\; k_{n_1}\Theta^{n_1 \cdots n_j}_{r_1 \cdots r_j} = 0$;

4) traceless: $\eta^{r_1 r_2}\Theta^{n_1 \cdots n_j}_{r_1 r_2 \cdots r_j} = 0$.

Instead of the tensor $\Theta^{n_1 \dots n_j}_{r_1 \dots r_j}(k)$ symmetrized in the upper and lower indices, it is convenient to consider the generating function

$$\Theta^{(j)}(x,y) = x^{r_1} \cdots x^{r_j}\, \Theta^{n_1 \dots n_j}_{r_1 \dots r_j}(k)\, y_{n_1} \cdots y_{n_j} \;.$$

D.Francia,J.Mourad,A.Sagnotti (2007); D.Ponomarev,A.Tseytlin (2016)

Proposition 5. The generating function $\Theta^{(j)}(x,y)$ of the covariant projection operator $\Theta^{n_1 \dots n_j}_{r_1 \dots r_j}$ (in D-dimensional space-time), satisfying properties 1)-4), listed in previous Proposition, has the form

$$\Theta^{(j)}(x,y) = \sum_{A=0}^{[\frac{j}{2}]} \Big(-\frac{1}{2}\Big)^A a^{(j)}_A \, \big(\Theta^{(y)}_{(y)}\, \Theta^{(x)}_{(x)}\big)^A \, \big(\Theta^{(y)}_{(x)}\big)^{j-2A} \;,$$

where $[\frac{j}{2}]$ – integer part of $\frac{j}{2}$, $\;a^{(j)}_0 = 1$, for $(A \geq 1)$ we have

$$a^{(j)}_A = \frac{j!}{(j-2A)!\, A!\, (2j+D-5)(2j+D-7)\cdots(2j+D-2A-3)} \;,$$

and the function $\Theta^{(y)}_{(x)}$ is defined as follows:

$$\Theta^{(y)}_{(x)} \equiv \Theta^{(1)}(x,y) = x^r\, y_n\, \Theta^n_r \;, \qquad \Theta^n_r = \delta^n_r - \frac{k_r k^n}{k^2} \;.$$

Remark 1.

The generating function $\Theta^{(j)}(x,y)$ satisfies differential equation

$$\frac{\partial}{\partial x^r}\frac{\partial}{\partial y_r}\Theta^{(j)}(x,y) = \frac{j(j+D-4)(2j+D-3)}{(2j+D-5)}\,\Theta^{(j-1)}(x,y) \;.$$

One can use this equation to calculate trace of the operator $\Theta^{(j)}$.

Remark 2.

The complete trace of the Behrends-Fronsdal projector $\Theta^{(j)}$ in the case of D-dimensional space-time ($D \geq 3$) is:

$$(\Theta^{(j)})^{r_1 r_2 \ldots r_j}_{r_1 r_2 \ldots r_j} = \frac{(D-4+j)!}{j!\,(D-3)!}\,(2j+D-3)\;.$$

This trace is equal to the dimension of the subspace, which is extracted from the space of vector-tensor wave functions $f_{n_1 \ldots n_j}(k)$ by the projector $\Theta^{(j)}$. In other words, this trace is equal to the number of independent components of symmetric wave functions $f_{(n_1 \ldots n_j)}(k)$:

$$k^{n_1} f_{(n_1 \ldots n_j)}(k) = 0\;, \quad \eta^{n_1 n_2} f_{(n_1 n_2 \ldots n_j)}(k) = 0\;.$$

On the space of these functions an irreducible massive representation of the D-dimensional rotation group is realized:

$$f_{i_1 \ldots i_r} = \boxed{i_1}\boxed{i_2}\boxed{\;\cdots\;}\boxed{i_r}\;.$$

Remark 3.

The naive construction for the projection operator of arbitrary representation of group $O(D)$ related to the Young diagram $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_k] \vdash j \in \mathbb{Z}_{\geq 0}$ is

$$\Theta^{(j)} = A(\mathsf{T}_\lambda) \cdot \Theta^{(\lambda_1)} \cdot \Theta^{(\lambda_2)} \cdots \Theta^{(\lambda_k)} \equiv Y(\mathsf{T}_\lambda) \cdot \Theta^{(\lambda_1)} \cdot \Theta^{(\lambda_2)} \cdots \Theta^{(\lambda_k)} ,$$

where $A(\mathsf{T}_\lambda)$ — antisymmetrizer "by columns" and $Y(\mathsf{T}_\lambda)$ — Young symmetrizer constructed with respect to Young tableaux T_λ. But this construction **is wrong** since the traces over indices from different columns are not equal to zero.

The correct way is to construct spin projectors to the invariant spaces (of irreps of $O(D)$) of symmetrized tensors (traceless and transverse) of rank j and related to the diagram $\lambda \vdash j$ is to construct all primitive orthogonal idempotents for the Brauer algebra $\mathcal{B}r_j$.

The Brauer algebra $\mathcal{B}r_j$ is generated by elements σ_k and κ_k $(k = 1, \dots, j-1)$ with defining relations

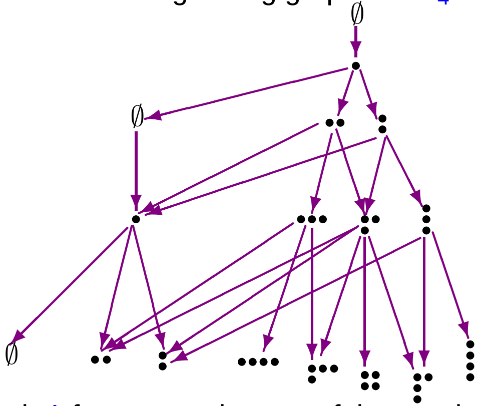
$$\sigma_i^2 = e \, , \quad \kappa_i^2 = \omega \, \kappa_i \, , \quad \sigma_i \kappa_i = \kappa_i \sigma_i = \kappa_i \, , \quad i = 1, \dots, n-1,$$
$$\sigma_i \sigma_j = \sigma_j \sigma_i \, , \quad \kappa_i \kappa_j = \kappa_j \kappa_i \, , \quad \sigma_i \, \kappa_j = \kappa_j \, \sigma_i \, , \quad |i-j| > 1,$$
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \, , \quad \kappa_i \kappa_{i+1} \kappa_i = \kappa_i \, , \quad \kappa_{i+1} \kappa_i \kappa_{i+1} = \kappa_{i+1} \, ,$$
$$\sigma_i \kappa_{i+1} \kappa_i = \sigma_{i+1} \kappa_i \, , \quad \kappa_{i+1} \kappa_i \sigma_{i+1} = \kappa_{i+1} \sigma_i \, , \quad i = 1, \dots, n-2 \, .$$

We need the representation s which acts in the space $(\mathbb{R}^D)^{\otimes j}$ of tensors of rank j (in this representation we have $\omega = D$)

$$s(\sigma_k) \cdot (\vec{e}_{i_1} \otimes \cdots \otimes \vec{e}_{i_k} \otimes \vec{e}_{i_{k+1}} \otimes \cdots \otimes \vec{e}_{i_n}) =$$
$$= (\vec{e}_{j_1} \otimes \cdots \otimes \vec{e}_{j_k} \otimes \vec{e}_{j_{k+1}} \otimes \cdots \otimes \vec{e}_{j_n}) \, \delta^{j_1}_{i_1} \cdots \delta^{j_{k-1}}_{i_{k-1}} \, \delta^{j_k}_{i_{k+1}} \, \delta^{j_{k+1}}_{i_k} \, \delta^{j_{k+2}}_{i_{k+2}} \cdots \delta^{j_n}_{i_n} \quad ,$$

$$s(\kappa_k) \cdot (\vec{e}_{i_1} \otimes \cdots \otimes \vec{e}_{i_k} \otimes \vec{e}_{i_{k+1}} \otimes \cdots \otimes \vec{e}_{i_n}) =$$
$$= (\vec{e}_{j_1} \otimes \cdots \otimes \vec{e}_{j_k} \otimes \vec{e}_{j_{k+1}} \otimes \cdots \otimes \vec{e}_{j_n}) \, \delta^{j_1}_{i_1} \cdots \delta^{j_{k-1}}_{i_{k-1}} \, \eta^{j_k j_{k+1}} \eta_{i_k i_{k+1}} \, \delta^{j_{k+2}}_{i_{k+2}} \cdots \delta^{j_n}_{i_n} \quad .$$

To construct all primitive orthogonal idempotents for the Brauer algebra $\mathcal{B}r_j$ one can use oscillating Young graph.
Example of oscillating Young graph for $\mathcal{B}r_4$:



Each path Λ from top to bottom of the graph corresponds to the primitive idempotent $e_\Lambda = \mathcal{B}r_4$. In the representation s the element e_Λ defines projector in $(\mathbb{R}^D)^{\otimes 4}$. E.g. the path $\{\emptyset \to \bullet \to \bullet\bullet \to \bullet\bullet\bullet \to \bullet\bullet\bullet\bullet\}$ corresponds to the symmetrizator of rank 4.

After construction of the idempotent $e_\Lambda \in \mathcal{B}r_j$ related to some Young diagram $\lambda \vdash j$ we need to find its image in the representation s and then substitute

$$\delta^i_j \;\to\; (\Theta^{(1)})^i_j = \delta^i_j - \frac{k^i k_j}{k^2}\,, \quad \eta^{ij} \;\to\; (\Theta^{(1)})^{ij} = \eta^{ij} - \frac{k^i k^j}{k^2} \qquad (\star)$$

After this, the projectors will have the transverse property.

Example. Symmetrizer

$$e_\Lambda = \frac{(y_2+1)\cdots(y_n+1)}{n!D^{n-1}}(y_2+D-1)\cdots(y_n+D-n+1)\,,$$

where Jucys-Merphy elements

$$y_1 = 0\,, \quad y_{m+1} = \sigma_m - \kappa_m + \sigma_m \cdot y_m \cdot \sigma_m\,,$$

form a maximal commutative subalgebra in $\mathcal{B}r_j$. For D-dimensional Behrends-Fronsdal projector we have

$$\Theta^{(j)} = \mathrm{s}(e_\Lambda)\,\big|_{(\star)}$$

Spin projection operator for half-integer spins

Remark 4. The following statement holds
Proposition 6. For arbitrary space-time dimension $D > 2$ and any half-integer spin j the projection operator $\Theta^{(j)}$ satisfies the conditions 1)-4) of the Proposition **3**, and the additional spinor condition

$$(\Theta^{(j)})^{n_1 \dots n_{j-1/2}}_{r_1 \dots r_{j-1/2}} \cdot \gamma_{n_1} = 0 = \gamma^{r_1} \cdot (\Theta^{(j)})^{n_1 \dots n_{j-1/2}}_{r_1 \dots r_{j-1/2}} \; ,$$

and the following formula holds:

$$((\Theta^{(j)})^{n_1 \dots n_{j-1/2}}_{r_1 \dots r_{j-1/2}})_A{}^B = c^{(j)} \; (\Theta^{(1/2)})_A{}^G \; (\gamma^r)_G{}^C (\gamma_n)_C{}^B (\Theta^{(j+\frac{1}{2})})^{n \, n_1 \dots n_{j-1/2}}_{r \, r_1 \dots r_{j-1/2}} \; ,$$

where $\Theta^{(j+\frac{1}{2})}$ – operator for the integer spin $(j+\frac{1}{2})$, factor $c^{(j)} = \frac{j+1/2}{(2j+D-2)}$ and $(\Theta^{(1/2)}) = \frac{1}{2\mathrm{m}}(\gamma^n k_n + \mathrm{m} \, I)$ (here operator I is $2^{[D/2]} \otimes 2^{[D/2]}$ unit matrix; $[a]$ denotes the integer part of a), matrices γ^n $(n = 0, 1, \cdots, D-1)$ represents generators of the Clifford algebra in D dimensions.

Conclusion.

We hope that the formalism considered here for describing massive particles of arbitrary spin will be useful in the construction of scattering amplitudes of massive particles in a similar way to the construction of spinor-helicity scattering amplitudes for massless particles. Some steps in this direction have already been done in papers:

1. E. Conde, E. Joung and K. Mkrtchyan, *Spinor-Helicity Three-Point Amplitudes from Local Cubic Interactions*, Journal of High Energy Physics 08 (2016) 040; arXiv:1605.07402 [hep-th].
2. A. Marzolla, *The 4D on-shell 3-point amplitude in spinor-helicity formalism and BCFW recursion relations*, in Proceedings of 12th Modave Summer School in Mathematical Physics (11-17 Sep 2016, Modave, Belgium), (2017) 002; arXiv:1705.09678 [hep-th].
3. N.Arkani-Hammed, T.C.Huang, Y.-t. Huang, *Scattering Amplitudes for all masses and spins* arXive:1709.04891[hep-th]