New deformations of extended supersymmetric mechanics

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“QFT & G”, July 30 - August 4, 2018, Tomsk

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Supersymmetric Quantum Mechanics (SQM) (Witten, 1981) is the simplest ($d = 1$) supersymmetric theory:

- Catches the basic features of higher-dimensional supersymmetric theories via the dimensional reduction;
- Provides superextensions of integrable models like Calogero-Moser systems, Landau-type models, etc;
- Extended $\mathcal{N} > 2, d = 1$ SUSY is specific: dualities between various supermultiplets, nonlinear “cousins” of off-shell linear multiplets, etc.
- $\mathcal{N} = 4$ SQM: $\{Q_\alpha, \bar{Q}^\beta\} = 2\delta_\alpha^\beta H\,, \alpha = 1, 2,$ is of special interest. In particular, a subclass of $\mathcal{N} = 4$ SQM models have as their bosonic target, Hyper-Kähler (HK) manifolds.

Deformed $\mathcal{N}=4$ SQM and its $\mathcal{N}=8$ extensions

The first type of deformed SQM arises, while choosing some semi-simple supergroups instead of higher-rank $d=1$ super-Poincare:

A. Standard extension:

$$(\mathcal{N}=2\,,\; d=1) \quad \Rightarrow \quad (\mathcal{N}>2\,,\; d=1 \text{ Poincaré})\,,$$

B. Non-standard extension:

$$(\mathcal{N}=2\,,\; d=1) \equiv u(1|1) \quad \Rightarrow \quad su(2|1) \;\subset\; su(2|2) \;\subset\; \ldots\,.$$

In the case B, the closure of supercharges contains, besides H, also internal symmetry generators. The deformed $\mathcal{N}=4$ SQM is associated with $su(2|1)$:

$$\{Q^i, \bar{Q}_j\} = 2m\left(I^i_j - \delta^i_j F\right) + 2\delta^i_j H\,, \qquad \left[I^i_j, I^k_l\right] = \delta^k_j I^i_l - \delta^i_l I^k_j\,,$$
$$\left[I^i_j, \bar{Q}_l\right] = \frac{1}{2}\delta^i_j \bar{Q}_l - \delta^i_l \bar{Q}_j\,, \qquad \left[I^i_j, Q^k\right] = \delta^k_j Q^i - \frac{1}{2}\delta^i_j Q^k,$$
$$\left[F, \bar{Q}_l\right] = -\frac{1}{2}\bar{Q}_l\,, \qquad \left[F, Q^k\right] = \frac{1}{2}Q^k.$$

The parameter m is a deformation parameter: when $m \to 0$, the standard $\mathcal{N}=4, d=1$ super-Poincare is restored.

- The simplest models with world-line realization of $su(2|1)$ were considered in (Belucci & Nersessian, 2003, 2004; Smilga, 2004; Römelsberger, 2006, 2007).
- The systematic superfield approach to $su(2|1)$ supersymmetry was worked out in (I. & Sidorov, 2014, 2016; I., Sidorov & Toppan, 2015).
- Recently, $su(2|1)$ invariant versions of super Calogero-Moser systems were constructed and quantized (Fedoruk & I., 2017; Fedoruk, I., Lechtenfeld & Sidorov, 2017).

The common features of all these models are:

- The oscillator-type Lagrangians for the bosonic fields, with m^2 as the oscillator strength.
- The appearance of the Wess-Zumino type terms for the bosonic fields, of the type $\sim im(\dot{z}\bar{z} - z\dot{\bar{z}})$.
- At the lowest energy levels, wave functions form atypical $su(2|1)$ multiplets, with unequal numbers of the bosonic and fermionic states.

Deformed $\mathcal{N} = 8$ mechanics

The flat $\mathcal{N} = 8$ superalgebra,

$$\{Q_A, Q_B\} = 2\delta_{AB} H\,, \quad A, B = 1, \ldots, 8\,,$$

admits two deformations:

A. Superalgebra $su(2|2)$ (I., Lechtenfeld, Sidorov, 2016):

$$\left\{Q^{ia}, S^{jb}\right\} = 2im\left(\varepsilon^{ab} L^{ij} - \varepsilon^{ij} R^{ab}\right) + 2\,\varepsilon^{ab}\varepsilon^{ij} C,$$

$$\left\{Q^{ia}, Q^{jb}\right\} = 2\,\varepsilon^{ij}\varepsilon^{ab}(H + C_1), \qquad \left\{S^{ia}, S^{jb}\right\} = 2\,\varepsilon^{ij}\varepsilon^{ab}(H - C_1).$$

B. Superalgebra $su(4|1)$ (I., Lechtenfeld, Sidorov, 2018):

$$\left\{Q^I, \bar{Q}_J\right\} = 2m\,L^I_J + 2\delta^I_J \mathcal{H}, \quad I, J = 1, \ldots, 4\,,$$

$$\left[\mathcal{H}, Q^K\right] = -\,\frac{3m}{4}\,Q^K, \qquad \left[\mathcal{H}, \bar{Q}_L\right] = \frac{3m}{4}\,\bar{Q}_L\,.$$

$SU(4)$ automorphisms (instead of $SO(8)$ or $SU(2) \times SU(2)$).

Detailed exposition of $SU(4|1)$ mechanics will be given by Stepan Sidorov.

QK $\mathcal{N} = 4$ SQM as a deformation of HK SQM models

Another type of deformations of $\mathcal{N} = 4$ SQM models proceeds from the general Hyper-Kähler (HK) subclass of the latter. The deforrmed models are $\mathcal{N} = 4$ supersymetrization of the Quaternion-Kähler (QK) $d = 1$ sigma models (I. & Mezincescu, 2017).

Both HK and QK $\mathcal{N} = 4$ SQM models can be derived from $\mathcal{N} = 4, d = 1$ harmonic superspace approach (I. & Lechtenfeld, 2003).

HK manifolds are bosonic targets of sigma models with **rigid** $\mathcal{N} = 2, d = 4$ SUSY (Alvarez-Gaume, Freedman, 1980, 1981). After coupling these models to **local** $\mathcal{N} = 2, d = 4$ SUSY in the supergravity framework the target spaces are deformed into the so called Quaternion-Kähler (QK) manifolds (Bagger, Witten, 1983).

Both types of the manifolds are $4n$ dimensional, but their holonomy groups are in $Sp(n)$ and $Sp(1) \times Sp(n)$, respectively.

Harmonic $\mathcal{N} = 4, d = 1$ superspace

- Ordinary $\mathcal{N} = 4, d = 1$ superspace:

$$(t, \theta^i, \bar{\theta}_k), \quad i, k, = 1, 2;$$

- Harmonic extension:

$$(t, \theta^i, \bar{\theta}_k) \quad \Rightarrow \quad (t, \theta^i, \bar{\theta}_k, u_j^{\pm}),\; u^{+i} u_i^{-} = 1,\; u_i^{\pm} \in SU(2)_{Aut}.$$

- Analytic basis:

$$(t_A, \theta^+, \bar{\theta}^+, u_k^{\pm}, \theta^-, \bar{\theta}^-) \equiv (\zeta, u^{\pm}, \theta^-, \bar{\theta}^-)$$

$$\theta^{\pm} = \theta^i u_i^{\pm},\; \bar{\theta}^{\pm} = \bar{\theta}^k u_k^{\pm},\;\; t_A = t + i(\theta^+ \bar{\theta}^- + \theta^- \bar{\theta}^+)$$

- Analytic superspace and superfields:

$$D^+ = \frac{\partial}{\partial \theta^-},\; \bar{D}^+ = -\frac{\partial}{\partial \bar{\theta}^-}, \quad D^+ \Phi = \bar{D}^+ \Phi = 0 \;\Rightarrow\; \Phi = \Phi(\zeta, u^{\pm})$$

- Harmonic derivatives:

$$D^{\pm\pm} = u_\alpha^{\pm} \frac{\partial}{\partial u_\alpha^{\mp}} + \theta^{\pm} \frac{\partial}{\partial \theta^{\mp}} + \bar{\theta}^{\pm} \frac{\partial}{\partial \bar{\theta}^{\mp}} + 2i \theta^{\pm} \bar{\theta}^{\pm} \frac{\partial}{\partial t_{\mathrm{A}}},$$

$[D^+, D^{++}] = [\bar{D}^+, D^{++}] = 0 \quad \Rightarrow \quad D^{++} \Phi(\zeta, u^{\pm})$ is analytic

Basic $\mathcal{N}=4, d=1$ multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$

- Described off-shell by an analytic superfield $q^{+a}(\zeta,u)$:

$$(4,4,0) \qquad \Longleftrightarrow \qquad q^{+a}(\zeta,u) \propto (f^{ia},\chi^a,\bar\chi^a),\;\; a=1,2,$$

$$(a)\; D^+q^{+a}=0\;(Grassmann\;\;analyticity),$$
$$(b)\; D^{++}q^{+a}=0\;(Harmonic\;\;analyticity),$$

$$(a)+(b) \quad \Longrightarrow \quad q^{+a}=f^{ka}u^+_k+\theta^+\chi^a-\bar\theta^+\bar\chi^a-2i\theta^+\bar\theta^+\dot f^{ka}u^-_k\,.$$

- Free off-shell action:

$$S_{free}\sim\int dtd^4\theta du\, q^{+a}q^-_a\sim\int dt\left(\dot f^{ia}\dot f_{ia}-\frac{i}{2}\bar\chi^a\dot\chi_a\right),\quad q^{-a}:=D^{--}q^{+a}$$

- Nonlinear $d=1$ sigma model action:

$$S_{free}\sim\int dtd^4\theta du\,\mathcal{L}(q^{+a},q^{-b},u^\pm).$$

- In bosonic sector: HKT (“Hyper-Kähler with torsion”) sigma model. In components, the torsion appears in a term quartic in fermions.

How to construct general HK $\mathcal{N}=4, d=1$ sigma models? No torsion in this case, the geometry involves only Riemann curvature tensor. The answer was given in Delduc, I., 2010.

- The basic superfields are real analytic, $q^{+A}(\zeta, u) = f^{iA}u_i^+ + \ldots, i = 1, 2,\ A = 1, \ldots 2n$, it encompasses just $4n$ fields $f^{iA}(t)$ parametrizing the target bosonic manifold, $\widetilde{(q_A^+)} = \Omega^{AB} q_B^+$, with $\Omega^{AB} = -\Omega^{BA}$ a constant symplectic metric.

- The linear constraint $D^{++}q^{+A} = 0$ is promoted to a nonlinear one
$$D^{++}q^{+A} = \Omega^{AB}\frac{\partial L^{+4}(q^{+C}, u^{\pm})}{\partial q^{+B}}.$$

- The superfield action is bilinear as in the free case,
$$S_{HK} \sim \int dt d^4\theta du\, \Omega^{AB} q_B^+ q_A^- = \int dt \left[g_{iA\,kB}(f)\dot{f}^{iA}\dot{f}^{kB} + \ldots\right],$$
the whole interaction appears only on account of nonlinear deformation of the q^{+A}-constraint.

- L^{+4} is an analytic hyper-Kähler potential (Galperin, I., Ogivetsky, Sokatchev, 1986): every L^{+4} produces the component HK metric $g_{iA\,kB}(f)$ and, *vice versa*, each HK metric originates from some HK potential L^{+4}.

From $\mathcal{N}=4$ HK SQM to its QK deformation

The harmonic superspace approach supplies the most natural arena for defining $\mathcal{N}=4$ QK SQM. Basic new features of these models as compared to their HK prototypes are as follows.

1. QK SQM model corresponding to $4n$ dimensional QK manifold requires $n+1$ multiplets $(4,4,0)$ described by analytic superfields $q^{+a}(\zeta, w^{\pm}), (a=1,2), Q^{+r}(\zeta, w^{\pm}), (r=1,\ldots 2n)$. An extra superfield $q^{+a}(\zeta, w^{\pm})$ is $d=1$ analog of $\mathcal{N}=2, d=4$ "conformal compensator".

2. QK SQM actions are invariant under local $\mathcal{N}=4, d=1$ supersymmetry realized by the appropriate transformations of super coordinates, including harmonic variables $w_i^{\pm}$.

3. For ensuring local invariance it is necessary to introduce a supervielbein $E(\zeta, \theta^-, \bar{\theta}^-, w^{\pm})$ which is a general $\mathcal{N}=4, d=1$ superfield.

4. Besides the (q^+, Q^+) superfield part, the correct action should contain a "comological term" involving the vielbein superfield only.

Minimal local $\mathcal{N} = 4, d = 1$ SUSY

By analogy with the $\mathcal{N} = 2, d = 4$ case we postulate that local $\mathcal{N} = 4, d = 1$ SUSY preserves the Grassmann analyticity,

$$\delta t = \Lambda(\zeta, w)\,,\ \delta\theta^+ = \Lambda^+(\zeta, w)\,,\ \delta\bar\theta^+ = \bar\Lambda^+(\zeta, w)\,,$$
$$\delta w_i^+ = \Lambda^{++}(\zeta, w) w_i^-\,,\ \delta w_i^- = 0\,,$$
$$\delta\theta^- = \Lambda^-(\zeta, w, \theta^-, \bar\theta^-)\,, \quad \delta\bar\theta^- = \bar\Lambda^-(\zeta, w, \theta^-, \bar\theta^-)\,,$$

The explicit structure of the minimal set of analytic parameters is as follows

$$\begin{aligned} \Lambda &= 2b + \ldots \\ \Lambda^+ &= \lambda^i w_i^+ + \ldots \\ \Lambda^{++} &= \tau^{(ik)} w_i^+ w_k^+ + \ldots \\ \Lambda^- &= \lambda^i w_i^- + \ldots \end{aligned}$$

Here, $b(t)$, $\tau^{(ik)}(t)$ and $\lambda^i(t)$, $\bar\lambda^i(t)$ are arbitrary local parameters, bosonic and fermionic, respectively. The local $\mathcal{N} = 4, d = 1$ supergroup obtained is isomorphic to the classical (having no central charges) “small” $\mathcal{N} = 4$ superconformal symmetry.

How to generalize the $(4,4,0)$ superfields $q^{+A}(\zeta, w)$ to local SUSY?

- The simplest possibility is to keep the linear constraint

$$D^{++}q^{+a} = 0\,, \quad D^{++}Q^{+r} = 0\,.$$

- It is covariant under the transformations

$$\delta D^{++} = -\Lambda^{++}D^0\,,\ \delta q^{+a} = \Lambda_0\, q^{+a}\,,\ \delta Q^{+r} = \Lambda_0\, Q^{+r}\,,$$
$$\Lambda^{++} = D^{++}\Lambda_0\,.$$

- To construct invariant actions, one needs the transformations of the integration measures $\mu_H := dtdwd^2\theta^+ d^2\theta^-\,,\ \mu^{(-2)} := dtdwd^2\theta^+\,,$

$$\delta\mu^{(-2)} = 0\,, \quad \delta\mu_H = \mu_H\, 2\Lambda_0\,,$$

  and that of harmonic derivative D^{--},

$$\delta D^{--} = -(D^{--}\Lambda^{++})D^{--}\,.$$

Simplest invariant action

The basic part of the total invariant action of the analytic superfields $q^{+a}(\zeta, w)$, $a = 1, 2$, and $Q^{+r}(\zeta, w)$, $r = 1, 2, \ldots 2n$, can be written as

$$S_{(2)} = \int \mu_H \, E \, \mathcal{L}_{(2)}(q, Q), \quad \mathcal{L}_{(2)}(q, Q) = \gamma q^{+a} q^-_a - Q^{+r} Q^-_r \,,$$
$$q^-_a := D^{--} q^+_a \,, \quad Q^-_r := D^{--} Q^+_r \,,$$

and $\gamma = \pm 1$. The new object is vielbein E which is harmonic-independent, $D^{++}E = D^{--}E = 0$, and transforms as

$$\delta E = (-4\Lambda_0 + 2D^{--}\Lambda^{++})E \,, \qquad D^{++}(-4\Lambda_0 + 2D^{--}\Lambda^{++}) = 0 \,.$$

One more important term in the action is the “cosmological term”:

$$S_\beta = \beta \int \mu_H \sqrt{E} \,, \quad \delta S_\beta = \beta \int \mu_H \, D^{--}\Lambda^{++} \sqrt{E} = 0 \,.$$

The simplest locally $\mathcal{N}=4$ supersymmetric action so reads

$$S_{HP} \sim S_{(2)} + S_\beta = \int \mu_H \big[E\, \mathcal{L}_{(2)} + \beta\sqrt{E} \big].$$

Why should the "cosmological" term S_β be added?

$$E = E_{bos} + E_{ferm}\,,$$

$$E_{bos} = e + \theta^+\theta^- M - \bar\theta^+\bar\theta^-\bar M + \theta^+\bar\theta^-(\mu - i\dot e) + \bar\theta^+\theta^-(\mu + i\dot e)$$
$$+\,4i(\theta^+\bar\theta^+ w_i^- w_k^- - \theta^+\bar\theta^- w_i^- w_k^+ - \theta^-\bar\theta^+ w_i^- w_k^+ + \theta^-\bar\theta^- w_i^+ w_k^+)L^{(ik)}$$
$$+\,4\theta^+\bar\theta^+\theta^-\bar\theta^-[D + 2\dot L^{(ik)} w_i^+ w_k^-],$$

$$E_{ferm} = (\theta^- w_i^+ - \theta^+ w_i^-)\phi^i - (\bar\theta^- w_i^+ - \bar\theta^+ w_i^-)\bar\phi^i + 4i\theta^-\bar\theta^-(\theta^+ w_i^+\sigma^i - \bar\theta^+ w_i^+\bar\sigma^i)$$
$$+\,2i\theta^+\bar\theta^+[\theta^- w_i^-(2\sigma^i - \dot\phi^i) - \bar\theta^- w_i^-(2\bar\sigma^i - \dot{\bar\phi}^i)]\,.$$

The fields $(e, \phi^i, \bar\phi^i, L^{ik})$ are gauge fields of $\mathcal{N}=4, d=1$ supergravity, $(D, \sigma^i, \bar\sigma^i)$ are additional fields, Lagrange multipliers. So we deal with a kind of **non-minimal** $\mathcal{N}=4, d=1$ supergravity in the present case.

- Pass to the bosonic limit:

$$q^{+a} \Rightarrow f^{ia} w_i^+ - 2i\theta^+\bar\theta^+ \dot f^{ia} w_i^- \,,\; Q^{+r} \Rightarrow F^{ir} w_i^+ - 2i\theta^+\bar\theta^+ \dot F^{ir} w_i^- \,,\; E \Rightarrow E_{bos}$$

- In this limit,

$$L_{HP} \Rightarrow \frac{1}{2} e \left(\dot F^{ir}\dot F_{ir} - \gamma\, \dot f^{ia}\dot f_{ia}\right) + L_{ik}\left[F^{(ir}\dot F_r^{k)} - \gamma f^{(ia}\dot f_a^{k)}\right]$$
$$+ \frac{1}{4} D\Big(\gamma f^{ia} f_{ia} - F^{ir}F_{ir} + \beta \frac{1}{\sqrt{e}}\Big)$$
$$+ \frac{\beta}{4}\,\frac{1}{e^{3/2}}\Big[L^{ik}L_{ik} - \frac{1}{8}\left(M\bar M + \mu^2 + \dot e^2\right)\Big].$$

- The auxiliary fields $M, \bar M$ and μ fully decouple and can be put equal to zero by their equations of motion. Also, $e(t)$ is an analog of $d = 1$ vierbein, so it is natural to choose the gauge

$$e = 1\,.$$

- Then the bosonic Lagrangian becomes

$$L_{HP} \Rightarrow \frac{1}{2}\left(\dot F^{ir}\dot F_{ir} - \gamma\, \dot f^{ia}\dot f_{ia}\right) + L_{ik}\left[F^{(ir}\dot F_r^{k)} - \gamma f^{(ia}\dot f_a^{k)}\right]$$
$$+ \frac{1}{4} D\Big(\gamma f^{ia} f_{ia} - F^{ir}F_{ir} + \beta\Big) + \frac{\beta}{4} L^{ik}L_{ik}.$$

- At $\beta \neq 0$ L^{ik} can be eliminated by its algebraic equation of motion, while D serves as the Lagrange multiplier giving rise to the constraint relating f^{ia} and F^{ir}:

$$L^{ik} = -2\frac{1}{\beta}\left[F^{(ir}\dot{F}_r^{k)} - \gamma f^{(ia}\dot{f}_a^{k)}\right], \quad \gamma f^{ia} f_{ia} - F^{ir} F_{ir} + \beta = 0\,.$$

- Assuming that f^{ia} starts with a constant (compensator!), one uses local $SU(2)$ freedom, $\delta f^{ia} = \tau^i{}_l f^{la}$, to gauge away the triplet from f^{ia},

$$f^{(ia)} = 0 \,\rightarrow\, f_a^i = \sqrt{2}\,\delta_a^i\,\omega\,.$$

- Then the constraint can be solved as

$$(a)\,\gamma = 1 \;\Rightarrow\; \beta < 0\,, \quad \omega = \frac{|\beta|^{1/2}}{2}\sqrt{1 + \frac{1}{|\beta|}F^2}\,,$$

$$(b)\,\gamma = -1 \;\Rightarrow\; \beta > 0\,, \quad \omega = \frac{\beta^{1/2}}{2}\sqrt{1 - \frac{1}{\beta}F^2}\,.$$

- The final form of the bosonic action for $\gamma = 1$ is

$$L_{HP} = \frac{1}{2}\Big[(\dot{F}\dot{F}) + \frac{2}{|\beta|}\,(F_{r(i}\dot{F}_{j)}{}^{r})(F_s^{(i}\,\dot{F}^{sj)}) - \frac{1}{|\beta|}\,\frac{1}{1+\frac{1}{|\beta|}F^2}(F\dot{F})(F\dot{F})\Big].$$

  The option $\gamma = -1$ is recovered by the replacement $|\beta| \to -|\beta|$.

- These actions describe $d = 1$ nonlinear sigma models on non-compact and compact maximally “flat” $4n$ dimensional QK manifolds, respectively:

$$\widetilde{\mathbb{HP}}^{\rm n} = \frac{Sp(1,n)}{Sp(1)\times Sp(n)}\,, \quad \mathbb{HP}^{\rm n} = \frac{Sp(1+n)}{Sp(1)\times Sp(n)}\,.$$

- Thus $\mathcal{N} = 4$ mechanics constructed is just superextensions of these QK $d = 1$ sigma models.

Generalizations

- The basic step in generalizing to $\mathcal{N}=4$ mechanics with an arbitrary QK manifold is to pass to nonlinear harmonic constraints

$$D^{++}q^{+a} - \gamma\frac{1}{2}\frac{\partial}{\partial q^+_a}\left[\hat{\kappa}^2(w^- \cdot q^+)^2\mathcal{L}^{+4}\right] = 0\,,$$
$$D^{++}Q^{+r} + \frac{1}{2}\frac{\partial}{\partial Q^+_r}\left[\hat{\kappa}^2(w^- \cdot q^+)^2\mathcal{L}^{+4}\right] = 0\,,$$
$$\mathcal{L}^{+4} \equiv \mathcal{L}^{+4}\Big(\frac{Q^{+r}}{\hat{\kappa}(w^-q^+)}, \frac{q^{+a}}{(w^-q^+)}, w^-_i\Big), \quad \hat{\kappa} := \frac{\sqrt{2}}{|\beta|^{1/2}}.$$

- The invariant superfield action is the same as in the $\mathbb{HP}^n$ case

$$\mathbb{S}_{QK} \sim \left[\tilde{S}_{(2)} + S_\beta\right] = \int \mu_H\Big[E\,\tilde{\mathcal{L}}_{(2)} + \beta\sqrt{E}\Big],$$
$$\tilde{\mathcal{L}}_{(2)} = \gamma q^{+a}q^-_a - Q^{+r}Q^-_r\,, \quad q^-_a = D^{--}q^{+a}\,, \quad Q^-_r = D^{--}Q^{+r}\,.$$

- The bosonic action precisely coincides with $d=1$ reduction of the general QK sigma model action derived from $\mathcal{N}=2, d=4$ supergravity-matter action in I., Valent, 2000. This coincidence proves that we have constructed most general QK $\mathcal{N}=4$ mechanics.

- One more possibility is to consider the following generalization of the $\mathbb{HP}^n$ action

$$S^{loc}(q,Q) = \int \mu_H \sqrt{E}\mathcal{F}(X,Y,w^-),\ X := \sqrt{E}\,(q^{+a}q^-_a)\,,\ Y := \sqrt{E}\,(Q^{+r}Q^-_r),$$
$$D^{++}q^{+a} = D^{++}Q^{+r} = 0 \quad \Rightarrow \quad D^{\pm\pm}X = D^{\pm\pm}Y = 0\,.$$

- When $E = const$, it is reduced to the particular form of the HKT action $\int \mu_H \mathcal{F}(q^{+A}, q^{-B}, w^{\pm})$, while for $\mathcal{F}(X,Y,w^-) = \gamma\,X - Y + \beta$ just to $\mathbb{HP}^n$ action. So the target geometry associated with $S^{loc}(q,Q)$ is expected to be a kind of QKT, i.e. “Quaternion-Kähler with torsion”. To date, not too much known about such geometries...

Summary and Outlook

- The self-consistent deformations of $\mathcal{N} = 8$ supersymmetric mechanics based on the supergroups $SU(2|2)$ and $SU(4|1)$ as a generalization of the $SU(2|1)$ mechanics exist.
- $\mathcal{N} = 4, d = 1$ harmonic superspace methods allow one to construct a new class of deformed $\mathcal{N} = 4$ supersymmetric mechanics models, those with $d = 1$ Quaternion-Kähler sigma models as a bosonic core. The basic distinguishing feature of these models is *local* $\mathcal{N} = 4, d = 1$ supersymmetry.
- The superfield and component actions are now known both for general $\mathcal{N} = 4$ QK mechanics, and for the maximally “flat” $\mathbb{HP}^n$ mechanics.
- A few generalizations of QK mechanics can be constructed, in particular “Quaternion-Kähler with torsion” (QKT) models.

- *Some further lines of study*:

(a) To construct the Hamiltonian formalism for the new class of $\mathcal{N} = 4$ mechanical systems, to perform quantization, at least for the simplest case of $\mathbb{HP}^{\rm n}$ mechanics, to find the energy spectra.

Last news (I & Mezincescu, 2018, in preparation): Noether currents were calculated and shown to be vanishing on-shell (like in the case of spinning particles, see, e.g., Pashnev & Sorokin, 1991),

$$Q^i = \bar{Q}^i = H = J_{kl} = 0\,.$$

(b) To explicitly construct some other $\mathcal{N} = 4$ QK SQM models, e.g. associated with symmetric QK manifolds (“Wolf spaces”).

(c) To construct locally supersymmetric versions of other off-shell $\mathcal{N} = 4, d = 1$ multiplets (such as $(3, 4, 1)$, $(1, 4, 3)$, etc) and of the associated SQM systems (Landau-type, Calogero-Moser-type and others).

(d) Links between the two types of SQM deformations?

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