Gravitational wave-like exact solutions in spatially homogeneous models of spacetime

Konstantin Osetrin

Tomsk State Pedagogical University

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Statement of the problem

Let us formulate the problem of finding models of a spatially-homogeneous spacetime that allow wave-like solutions. We assume that:

- the metric of spacetime is a plane-wave metric, that is, there exists a coordinate system where the metric depends only on one null wave variable;
- the spacetime under consideration is spatially homogeneous, that is, there exists a subgroup of spacetime isometries with 3-dimensional space-like orbits;
- the metric of spacetime satisfies the equations of the theory of gravitation (Einstein's vacuum equations).

It is required to find the explicit form of the spacetime metric under all the conditions described.

Plane wave spacetimes metric

We will call the model of spacetime as a plane wave-like if there is a coordinate system with respect to which the metric depends only on one null variable

$$dS^2 = g_{ij}\, dx^i dx^j, \qquad g_{ij} = g_{ij}(x^0), \qquad i, j, k = 0...3; \qquad (1)$$

where x^0 is a null variable, i.e.

$$dS = 0 \text{ for } dx^0 \neq 0 \text{ and } dx^p = 0, \qquad p, q, r = 1...3.$$

This metric can be represented in the following general form:

$$dS^2 = 2dx^0dx^1 + g_{ab}(x^0)\left(dx^a + g^a(x^0)dx^1\right)\left(dx^b + g^b(x^0)dx^1\right),$$
$$a, b, c = 2, 3. \qquad (2)$$

Plane wave Einstein's spacetimes metric

Substituting the metric (2) in the Einstein's equations $R_{ab} = 0$ it turns out that g^a are constants and the metric can be represented as:

$$dS^2 = 2\,dx^0 dx^1 + g_{ab}(x^0)\,dx^a\,dx^b, \qquad a, b = 2, 3; \tag{3}$$

For the metric (3) there remains only one component of the Ricci tensor R_{00}, that does not vanish identically.

Below, we consider the metric of a plane gravitational wave (3) for the case when spacetime is spatially homogeneous.

Symmetries of plane wave-like models of spatially homogeneous spacetimes

The model of spacetime considered admits 3 commuting Killing vectors $Y^i_{(p)}$:

$$Y^i_{(0)} = (0,\, 1,\, 0,\, 0)\,, \tag{4}$$

$$Y^i_{(1)} = (0,\, 0,\, 1,\, 0)\,, \tag{5}$$

$$Y^i_{(2)} = (0,\, 0,\, 0,\, 1)\,, \tag{6}$$

The vector $Y^i_{(0)}$ is a null vector, and the vectors $Y^i_{(1)}$ and $Y^i_{(2)}$ are spacelike vectors.

The additional spacelike Killing vector

The additional Killing vector, which can provides the spatial homogeneity of the model, must belong to two basic types:

$$\text{Type A:} \quad Y^i_{(3)} = \left(-x^0,\, x^1,\, ax^2 + bx^3,\, \tilde{a}x^2 + \tilde{b}x^3\right), \tag{7}$$

$$\text{Type B:} \quad Y^i_{(3)} = \left(1,\, 1,\, ax^2 + bx^3,\, \tilde{a}x^2 + \tilde{b}x^3\right), \tag{8}$$

where a, $\tilde{a}$, b, $\tilde{b}$ are constants.
Then Killing vectors $Y^i_{(1)}$, $Y^i_{(2)}$, $Y^i_{(3)}$ can provide spatial homogeneity of the model.
As follows from Einstein's vacuum equations, type B leads to contradictions. Therefore, below only type A is considered.

Spatially homogeneous wave-like model class A1

The additional Killing vector for this class of spacetimes has the following form:

$$Y^i_{(3)} = \left(-x^0,\, x^1,\, \lambda_2 x^2,\, \lambda_3 x^3\right), \tag{9}$$

where x^0 is a null wave-like variable and λ_2, λ_2 – constants.

The commutation relations for the Killing vectors defining a subgroup of spatial isometry in the class A1 have the form:

$$\begin{aligned} \left[Y_{(1)}, Y_{(2)}\right] &= 0, & (10)\\ \left[Y_{(1)}, Y_{(3)}\right] &= \lambda_2 Y_{(1)}, & (11)\\ \left[Y_{(2)}, Y_{(3)}\right] &= \lambda_3 Y_{(2)}. & (12) \end{aligned}$$

Spatially homogeneous wave-like metric class A1

$$g_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{{x^0}^{2\lambda_2}}{\sigma^2} & -\frac{\alpha {x^0}^{\lambda_2+\lambda_3}}{\sigma^2} \\ 0 & 0 & -\frac{\alpha {x^0}^{\lambda_2+\lambda_3}}{\sigma^2} & \frac{{x^0}^{2\lambda_3}}{\sigma^2} \end{pmatrix}, \qquad (13)$$

where α, λ_2, λ_3 are constant parameters of model, x^0 is a null wave variable.

$$g = \det g_{ij} = -\frac{{x^0}^{2(\lambda_2+\lambda_3)}}{\sigma^2}, \qquad \sigma^2 = 1 - \alpha^2,$$

$$-1 < \alpha < 1, \qquad 0 < \sigma^2 \leq 1.$$

Spatially homogeneous wave-like solution class A1

In this case, the positive definiteness of the restriction of the spacetime metric to the orbits of the subgroup of isometry is determined by the one condition:

$$x^0 x^1 < 0. \tag{14}$$

The condition (14) can be provided by changing the coordinates

$$x^0 = \exp \tilde{x}^0, \qquad x^1 = -\exp(\pm \tilde{x}^1). \tag{15}$$

From Einstein's vacuum equations we have only one condition:

$$(\lambda_2 - \lambda_3)^2 + \sigma^2(\lambda_2 + \lambda_3 - 1)^2 = \sigma^2. \tag{16}$$

Then we get the following restrictions:

$$(\lambda_2 - \lambda_3)^2 \leq \sigma^2, \qquad -1 \leq (\lambda_2 - \lambda_3) \leq 1, \qquad 0 \leq (\lambda_2 + \lambda_3) \leq 2,$$

Spatially homogeneous wave-like solution class A1

One can express λ_2 by λ_3:

$$\lambda_2 = \frac{\lambda_3 + \sigma^2(1-\lambda_3) \pm \sqrt{\sigma^4 + 4\sigma^2\lambda_3(1-\lambda_3)}}{\sigma^2+1}.$$

Then in this model there are two independent constant parameters of three α, λ_2, λ_3 ($\sigma^2 = 1-\alpha^2$) and

$$-1 < \alpha < 1, \qquad \frac{1-\sqrt{2}}{2} \leq \lambda_{2,3} \leq \frac{1+\sqrt{2}}{2},$$

more exactly

$$\frac{1}{2}\left(1-\sqrt{1+\sigma^2}\right) \leq \lambda_{2,3} \leq \frac{1}{2}\left(1+\sqrt{1+\sigma^2}\right).$$

Weyl tensor of spatially homogeneous wave-like model A1

In the general case, there remain three independent non-zero components of the Weyl tensor C_{0202}, C_{0303} and C_{0203}:

$$\mathrm{C}_{0202} = \frac{(\lambda_2 - \lambda_3)\Big(1 - \lambda_2 - \lambda_3 + \alpha^2(2\lambda_2 - 1)\Big)x^{0\,2(\lambda_2 - 1)}}{2(1-\alpha^2)^2},$$

$$\mathrm{C}_{0203} = -\frac{\alpha(\lambda_2 - \lambda_3)^2 x^{0\,(\lambda_2+\lambda_3-2)}}{2(1-\alpha^2)^2},$$

$$\mathrm{C}_{0303} = -\frac{(\lambda_2 - \lambda_3)\Big(1 - \lambda_2 - \lambda_3 + \alpha^2(2\lambda_3 - 1)\Big)x^{0\,2(\lambda_3 - 1)}}{2(1-\alpha^2)^2},$$

Note that for each value of $\alpha \neq 0$ there are 2 pairs of values of λ_2, λ_3 for which C_{0202} can be turned to zero, while C_{0303} will not be zero and vice versa. For example: for $\alpha = \pm 1/\sqrt{2}$, $\lambda_2 = 1/2$ and $\lambda_3 = (3 \pm 2\sqrt{3})/6$ we have $\mathrm{C}_{0303} = 0$, but $\mathrm{C}_{0202} \neq 0$.

Flat solutions of class A1

Spacetime A1 class becomes flat in the following cases only:

1. $\lambda_2 = \lambda_3 \quad (\text{then } \lambda_2 = \lambda_3 = 0 \text{ or } 1);$ (17)
2. $\alpha = 0, \quad \lambda_2 = 0, \quad \lambda_3 = 1;$ (18)
3. $\alpha = 0, \quad \lambda_2 = 1, \quad \lambda_3 = 0.$ (19)

Non-flat spacetimes of class A1 are of type VI_a according to Bianchi's classification and type N according to Petrov's classification.

Spatially homogeneous wave-like models class A2

The additional Killing vector for this class of spacetimes has the following form:

$$Y^i_{(3)} = \left(-x^0,\, x^1,\, \lambda x^2,\, x^2 + \lambda x^3\right), \qquad (20)$$

where x^0 is a null wave-like variable, λ is a constant.
The commutation relations for the class A2 have the form:

$$\left[Y_{(1)}, Y_{(2)}\right] = 0, \qquad (21)$$
$$\left[Y_{(1)}, Y_{(3)}\right] = \lambda Y_{(1)} + Y_{(2)}, \qquad (22)$$
$$\left[Y_{(2)}, Y_{(3)}\right] = \lambda Y_{(2)}. \qquad (23)$$

Spatially homogeneous wave-like metric class A2

$$g_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{x^{0\,2\lambda}\left(\alpha^2 \ln^2 x^0 - 2\beta \ln x^0 + \gamma^2\right)}{\sigma^2} & \frac{x^{0\,2\lambda}(\alpha^2 \ln x^0 - \beta)}{\sigma^2} \\ 0 & 0 & \frac{x^{0\,2\lambda}(\alpha^2 \ln x^0 - \beta)}{\sigma^2} & \frac{\alpha^2 x^{0\,2\lambda}}{\sigma^2} \end{pmatrix} \qquad (24)$$

where α, β, γ, λ are constant parameters of model, x^0 is a null wave-like variable.

$$g = \det g_{ij} = -\frac{x^{0\,4\lambda}}{\sigma^2}, \qquad \sigma^2 = \alpha^2\gamma^2 - \beta^2,$$

$$\alpha\gamma\sigma \neq 0, \qquad 0 \leq \beta^2 < (\alpha\gamma)^2.$$

Spatially homogeneous wave-like solution class A2

From Einstein's equations we have one conditon only:

$$\lambda = \frac{\sigma^2 \pm \sqrt{\sigma^2(\sigma^2 - \alpha^4)}}{2\sigma^2}, \qquad 0 < \lambda < 1,$$

Then in this model there are three independent constant parameters α, β and γ ($\sigma^2 = \alpha^2\gamma^2 - \beta^2$), and

$$\beta^2 + \alpha^4 < \alpha^2\gamma^2, \qquad \alpha^4 < \sigma^2, \qquad \alpha^2 < \gamma^2.$$

Spatially homogeneous wave-like solution class A2

In this case, the positive definiteness of the restriction of the spacetime metric to the orbits of the subgroup of isometry is determined by the one condition:

$$x^0 x^1 < 0. \tag{25}$$

The condition (25) can be provided by changing the coordinates:

$$x^0 = \exp \tilde{x}^0, \qquad x^1 = -\exp(\pm \tilde{x}^1). \tag{26}$$

Weyl tensor of spatially homogeneous wave-like model A2

Non-zero components of the Weyl tensor C_{ijkl}:

$$\mathrm{C}_{0202} = {x^0}^{2\lambda}\Big(-2\alpha^2 \ln(x^0)\left(\sigma^2(2\lambda-1)+\alpha^2\gamma\right)$$

$$+\alpha^4\gamma^2 - 2\sigma^2(\alpha^2-2\gamma\lambda+\gamma)+\alpha^6\ln^2 x^0\Big),$$

$$\mathrm{C}_{0203} = \alpha^2 {x^0}^{2\lambda}\Big(\sigma^2(1-2\lambda)-\alpha^2\gamma+\alpha^4\ln x^0\Big),$$

$$\mathrm{C}_{0303} = \alpha^6 {x^0}^{2\lambda}, \qquad \alpha\gamma\sigma \neq 0.$$

Rieman tensor and Weyl tensor of the model can not be equal to 0. The spacetimes of the class A2 are of type IV according to Bianchi's classification and type N according to Petrov's classification.

Spatially homogeneous wave-like model class A3

The additional Killing vector for this class of spacetimes has the following form:

$$Y^i_{(3)} = \left(-x^0,\, x^1,\, \lambda x^2 - x^3,\, x^2 + \lambda x^3\right), \tag{27}$$

where x^0 is a null wave-like variable, λ is a constant.
The commutation relations for the class A3 have the form:

$$\left[Y_{(1)}, Y_{(2)}\right] = 0, \tag{28}$$
$$\left[Y_{(1)}, Y_{(3)}\right] = \lambda Y_{(1)} + Y_{(2)}, \tag{29}$$
$$\left[Y_{(2)}, Y_{(3)}\right] = -Y_{(1)} + \lambda Y_{(2)}. \tag{30}$$

Spatially homogeneous wave-like metric class A3

The metric tensor g_{ij} have a form:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{x^{0\,2\lambda}\left(\gamma-\alpha\cos(\ln x^{0\,2})-\beta\sin(\ln x^{0\,2})\right)}{\sigma^2} & \frac{x^{0\,2\lambda}\left(\alpha\sin(\ln x^{0\,2})-\beta\cos(\ln x^{0\,2})\right)}{\sigma^2} \\ 0 & 0 & \frac{x^{0\,2\lambda}\left(\alpha\sin(\ln x^{0\,2})-\beta\cos(\ln x^{0\,2})\right)}{\sigma^2} & \frac{x^{0\,2\lambda}\left(\gamma+\alpha\cos(\ln x^{0\,2})+\beta\sin(\ln x^{0\,2})\right)}{\sigma^2} \end{pmatrix}$$

where α, β, γ, λ are constant parameters of the model, x^0 is a null wave-like variable.

$$g = \det g_{ij} = -\frac{(x^0)^{4\lambda}}{\sigma^2}, \qquad \sigma^2 = \gamma^2 - \alpha^2 - \beta^2, \qquad \gamma \neq 0.$$

Spatially homogeneous wave-like solution class A3

From vacuum Einstein's equations we have one condition only:

$$\lambda = \frac{\sigma \pm \sqrt{\sigma^2 - 4(\alpha^2+\beta^2)}}{2\sigma}, \qquad 0 \leq \lambda \leq 1, \tag{31}$$

Then in this model there are three independent constant parameters α, β and γ ($\sigma = \pm\sqrt{\gamma^2-\alpha^2-\beta^2}$), and

$$4(\alpha^2+\beta^2) \leq \sigma^2, \tag{32}$$

$$5\,(\alpha^2+\beta^2) \leq \gamma^2. \tag{33}$$

Spatially homogeneous wave-like solution class A3

The positive definiteness of the restriction of the spacetime metric to the orbits of the subgroup of isometry is determined by the conditions:

$$x^0 x^1 < 0, \qquad 0 < \gamma + \alpha \cos(\ln {x^0}^2) + \beta \sin(\ln {x^0}^2). \qquad (34)$$

The conditions (34) taking into account previous conditions (33) (from Einstein's equations) can be provided by:

$$x^0 = \exp \tilde{x}^0, \qquad x^1 = -\exp(\pm \tilde{x}^1), \qquad 0 < \gamma. \qquad (35)$$

Weyl tensor of spatially homogeneous wave-like model A3

Non-zero components of the Weyl tensor C_{ijkl}:

$$\mathrm{C}_{0202} = -{x^0}^{2\lambda}\Big(\cos(2\ln x^0)\big(2\alpha^3 + 2\alpha\left(\beta^2+\sigma^2\right) + \beta(1-2\lambda)\sigma^2\big)$$

$$-2\gamma\left(\alpha^2+\beta^2\right) + \sin(2\ln x^0)\big(2\alpha^2\beta + \alpha(2\lambda-1)\sigma^2 + 2\beta\left(\beta^2+\sigma^2\right)\big)\Big),$$

$$\mathrm{C}_{0203} = {x^0}^{2\lambda}\Big(\sin(2\ln x^0)\left(2\alpha^3 + 2\alpha\left(\beta^2+\sigma^2\right) + \beta(1-2\lambda)\sigma^2\right)$$

$$-\cos(2\ln x^0)\left(2\alpha^2\beta + \alpha(2\lambda-1)\sigma^2 + 2\beta\left(\beta^2+\sigma^2\right)\right)\Big),$$

$$\mathrm{C}_{0303} = {x^0}^{2\lambda}\Big(\cos(2\ln x^0)\big(2\alpha^3 + 2\alpha\left(\beta^2+\sigma^2\right) + \beta(1-2\lambda)\sigma^2\big)$$

$$+2\gamma\left(\alpha^2+\beta^2\right) + \sin(2\ln x^0)\big(2\alpha^2\beta + \alpha(2\lambda-1)\sigma^2 + 2\beta\left(\beta^2+\sigma^2\right)\big)\Big).$$

The model A3 becomes flat only if $\alpha = \beta = 0$ (then $\lambda = 0$ or 1).
The spacetimes A3 class are of type VII_a according to Bianchi's classification and type N according to Petrov's classification.

Conclusion

- A classification of spatially homogeneous plane-wave models of spacetime is constructed.
- Three classes of wave-like spatially homogeneous exact models of spacetime are obtained (A1, A2, A3).
- The models considered can describe the primordial gravitational waves of the Universe.
- The models considered can be used to obtain exact wave-like solutions in modified theories of gravity.