# ‘t Hooft-Polyakov monopole and disclinations in the geometric theory

M. Katanaev
Steklov Mathematical Institute, Moscow

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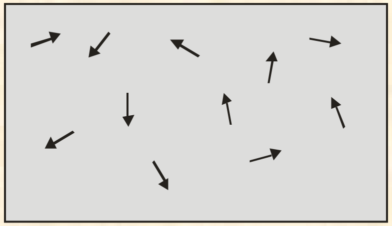
# Disclinations

Ferromagnets



$n^i(x)$ - unit vector field $\qquad n_0^i$ - fixed unit vector

$$n^i = n_0^j S_j{}^i(\omega)$$

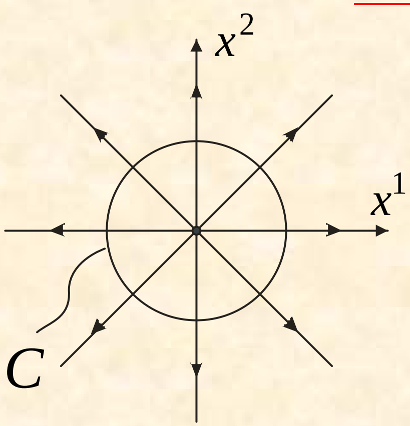
$S_i{}^j \in \mathbb{SO}(3)$ - orthogonal matrix

$\omega^{ij} = -\omega^{ji} \in \mathfrak{so}(3)$ - Lie algebra element (spin structure)

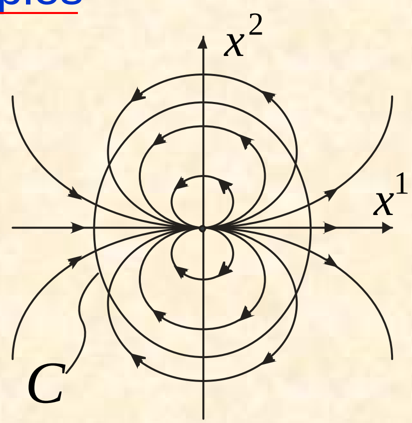
$\omega_i = \frac{1}{2}\varepsilon_{ijk}\omega^{jk}$ - rotational angle

$\varepsilon_{ijk}$ - totally antisymmetric tensor $(\varepsilon_{123} = 1)$

Examples




$\Theta = 2\pi$




$\Theta = 4\pi$

$$\Omega^{ij} = \oint_C dx^\mu \partial_\mu \omega^{ij}$$

$\Theta_i = \varepsilon_{ijk}\Omega^{jk}$ - Frank vector (total angle of rotation)

$$\Theta = \sqrt{\Theta^i \Theta_i}$$

# Frank vector

$\omega^{ij}(x)$ - is not continuous !

$$\omega_{\mu}{}^{ij}(x) = \begin{cases} \partial_{\mu}\omega^{ij} & \text{- outside the cut} \\ \lim \partial_{\mu}\omega^{ij} & \text{- on the cut} \end{cases}$$

- SO(3)-connection (continuous on the cut)

$$\Omega^{ij} = \oint dx^{\mu}\omega_{\mu}{}^{ij} = \iint dx^{\mu} \wedge dx^{\nu}(\partial_{\mu}\omega_{\nu}{}^{ij} - \partial_{\nu}\omega_{\mu}{}^{ij})$$ - the Frank vector

$$R_{\mu\nu}{}^{ij} = \partial_{\mu}\omega_{\nu}{}^{ij} - \omega_{\mu}{}^{ik}\omega_{\nu k}{}^{j} - (\mu \leftrightarrow \nu)$$ - curvature

$$\boxed{\Omega^{ij} = \iint dx^{\mu} \wedge dx^{\nu} R_{\mu\nu}{}^{ij}}$$ - definition of the Frank vector in the geometric theory

Back to the spin structure: if $n \in \mathbb{R}^2$ then $\mathbb{SO}(3) \rightarrow \mathbb{SO}(2)$

## Summary of the geometric approach

## (physical interpretation)

Media with dislocations and disclinations $=$

$= \mathbb{R}^3$ with a given Riemann-Cartan geometry

Independent variables $\begin{cases} e_\mu{}^i & \text{- triad field} \\ \omega_\mu{}^{ij} & \text{- SO(3)-connection} \end{cases}$

$$T_{\mu\nu}{}^i = \partial_\mu e_\nu{}^i - \omega_\mu{}^{ij} e_{\nu j} - (\mu \leftrightarrow \nu)$$

- torsion (surface density of the Burgers vector)

$$R_{\mu\nu}{}^{ij} = \partial_\mu \omega_\nu{}^{ij} - \omega_\mu{}^{ik} \omega_{\nu k}{}^j - (\mu \leftrightarrow \nu)$$

- curvature (surface density of the Frank vector)

Elastic deformations: $R_{\mu\nu}{}^{ij} = 0, \quad T_{\mu\nu}{}^i = 0$

Dislocations: $R_{\mu\nu}{}^{ij} = 0, \quad T_{\mu\nu}{}^i \neq 0$

Disclinations: $R_{\mu\nu}{}^{ij} \neq 0, \quad T_{\mu\nu}{}^i = 0$

Dislocations and disclinations: $R_{\mu\nu}{}^{ij} \neq 0, \quad T_{\mu\nu}{}^i \neq 0$

## $SO(3)$ - model

$(x^\alpha) \in \mathbb{R}^{1,3}, \;\; \alpha = 0,1,2,3$ - Minkowskian space-time

$n(x)$ - the basic variable

$$n = (n^i) \in \mathbb{S}^2, \quad i = 1,2,3, \quad n^2 := n^i n^j \delta_{ij} = 1$$

$SO(3)$ - model:

$$S := \int dx \frac{1}{2}(\partial n)^2 := \int dx \frac{1}{2}\eta^{\alpha\beta}\partial_\alpha n^i \partial_\beta n^j \delta_{ij}$$

$$n := (\sin\Theta\cos\Phi, \, \sin\Theta\sin\Phi, \, \cos\Theta)$$ - parameterization

$\Theta(x), \, \Phi(x)$ - azimuthal and polar angles in the target space

The Lagrangian:

$$L = \frac{1}{2}(\partial\Theta^2 + \sin^2\partial\Phi^2)$$

# Angular parameterization

$$n^i(x) := n_0^j S_j{}^i\big(\omega(x)\big), \qquad S_i{}^j \in \mathbb{O}(3)$$ - orthogonal matrix

$$\omega = (\omega^i) \in so(3) \approx \mathbb{R}^3$$ - element of the algebra

$$S_i{}^j = \delta_i^j \cos\omega + \frac{(\omega\varepsilon)_i{}^j}{\omega}\sin\omega + \frac{\omega_i\omega^j}{\omega^2}(1-\cos\omega)$$ - angular parameterization

$$\omega := \sqrt{\omega^i\omega_i}$$ - the length of a vector $\qquad (\omega\varepsilon)_i{}^j := \omega^k \varepsilon_{ki}{}^j$

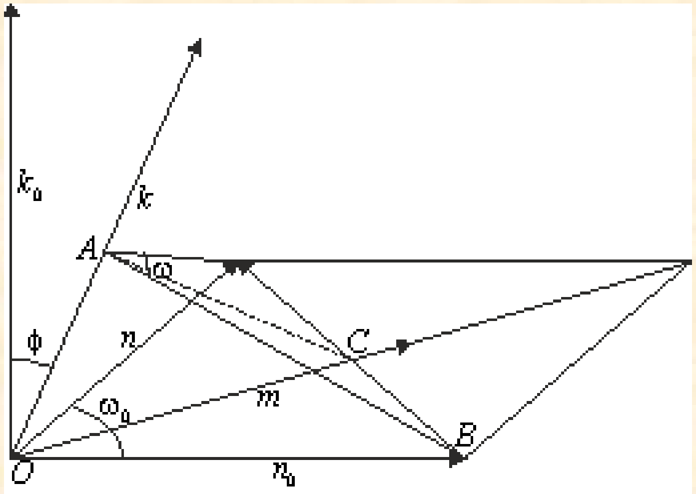
$$\omega^i \sim \omega^i + 2\pi\frac{\omega^i}{\omega}$$ - the equivalence relation

$$\{\omega^i\} \to \{k^i, \omega\}, \quad k^i := \frac{\omega^i}{\omega}$$ - new variables $\qquad k^2 = 1, \quad \omega \in [-\pi, \pi]$

$$S_i{}^j = \delta_i^j \cos\omega + k^k \varepsilon_{ki}{}^j \sin\omega + k_i k^j (1-\cos\omega)$$

$$n^i = n_0^j S_j{}^i(\omega, k) = n_0^j S_j{}^k(\omega, k) S_k{}^i(\psi, n)$$

$\psi(x)$ - arbitrary angle $\qquad \psi(x) \to \psi(x) + \alpha(x)$ - gauge transformation




## The Lagrangian

$$L = \frac{\partial\omega^2}{2}\left[1-\frac{\sin^2\omega}{\omega^2}-\frac{(n_0,\omega)^2}{\omega^2}\left(1-\frac{\sin\omega}{\omega}\right)^2\right]+$$

$$+\frac{(\partial^\alpha\omega,\partial_\alpha\omega)}{2\omega^2}\left[\sin^2\omega+\frac{4(n_0,\omega)^2}{\omega^2}\sin^4\frac{\omega}{2}\right]-\frac{2(n_0,\partial_\alpha\omega)^2}{\omega^2}\sin^2\frac{\omega}{2}\cos\omega-$$

$$-\frac{\partial^\alpha\omega(n_0,\partial_\alpha\omega)(n_0,\omega)}{\omega^2}\left(\sin\omega-\frac{4}{\omega}\sin^2\frac{\omega}{2}\cos\omega\right)-$$

$$-\frac{2\partial^\alpha\omega^i\omega^j n_0^k\varepsilon_{ijk}}{\omega^3}\sin^2\frac{\omega}{2}\left[(n_0,\partial_\alpha\omega)\sin\omega+\partial_\alpha\omega(n_0,\omega)\left(1-\frac{\sin\omega}{\omega}\right)\right]$$

$\omega^i(x)$ - independent variable

## Gauge transformation

$$\sin\omega' = \frac{2\sin\frac{\omega}{2}\sin\upsilon\left(\cos\frac{\omega}{2}\sin\upsilon\cos\alpha - \cos\upsilon\sin\alpha\right)}{1-\left(\cos\upsilon\cos\alpha + \cos\frac{\omega}{2}\sin\upsilon\sin\alpha\right)^2}$$

$$k'^i = k^i\cos\alpha + \left(-k^i\cos\frac{\omega}{2}\cos\upsilon + n_0^i\cos\frac{\omega}{2} + n_0^j k^k \varepsilon_{kj}{}^i \sin\frac{\omega}{2}\right)\frac{\sin\alpha}{\sin\upsilon}$$

$\alpha(x)$ - gauge parameter

$\cos\upsilon := (\mathrm{n}_0, k)$

$k^i := \frac{\omega^i}{\omega}$ - notation

# ‘t Hooft-Polyakov monopole

$(x^\alpha) \in \mathbb{R}^{1,3},\ \alpha = 0,1,2,3$ $\quad \eta_{\alpha\beta} := \mathrm{diag}(+---)$ - Lorentz metric

$i, j = 1,2,3$ $\quad \delta_{ij} := \mathrm{diag}(+++)$ - metric in target space

$$L = -\frac{1}{4} F^{\alpha\beta i} F_{\alpha\beta i} + \frac{1}{2} \nabla^\alpha \varphi^i \nabla_\alpha \varphi_i - \frac{1}{4} \lambda (\varphi^2 - a^2)^2$$

- the geometric model

$$F_{\alpha\beta}{}^i := \partial_\alpha A_\beta{}^i - \partial_\beta A_\alpha{}^i + e A_\alpha{}^j A_\beta{}^k \varepsilon_{jk}{}^i$$

$A_\alpha{}^i$ - $\mathbb{SU}(2)$ gauge field

$\varphi = (\varphi^i) \in \mathbb{R}^3$ - triplet of scalar fields in adjoint representation of $\mathbb{SU}(2)$ group

$\nabla_\alpha \varphi^i := \partial_\alpha \varphi^i + e A_\alpha{}^j \varphi^k \varepsilon_{jk}{}^i$ - covariant derivative

$e \in \mathbb{R},\ \lambda, a > 0$ - coupling constants

$$\mathbb{SO}(3) = \frac{\mathbb{SU}(2)}{\mathbb{Z}_2}$$

## Static solutions

$$A_\alpha{}^i = 0, \qquad \varphi^i = \text{const}, \qquad \varphi^2 = a^2$$ - vacuum solution

$$E = \frac{1}{4} F^{\mu\nu i} F_{\mu\nu i} + \frac{1}{2} \nabla^\mu \varphi^i \nabla_\mu \varphi_i + \frac{1}{4} \lambda (\varphi^2 - a^2)^2$$ - the energy

$(1+3)$ decomposition:

$$(x^\alpha) = (x^0, x^\mu) = (x^0, \boldsymbol{x}), \qquad (A_\alpha{}^i) := (A_0{}^i, A_\mu{}^i), \quad \mu := 1,2,3$$

$$A_\alpha{}^i = A_\alpha{}^i(\boldsymbol{x}), \quad \varphi^i(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^3$$ - static solutions

$$A_0{}^i = 0$$ - additional requirement

$$\nabla_\nu F^{\nu\mu}{}_i + e(\nabla^\mu \varphi^j) \varphi^k \varepsilon_{ikj} = 0$$

$$-\nabla^\mu \nabla_\mu \varphi_i - \lambda \left( \varphi^2 - a^2 \right) \varphi_i = 0$$

- equilibrium equations

$$A_0{}^i \Big|_{r=\infty} = 0, \qquad \varphi^2 \Big|_{r=\infty} = a^2$$ - boundary conditions

# Spherically symmetric solutions

Assumption: $\mathbb{SU}(2)$ acts simultaneously in coordinate and target spaces

$$A_\mu{}^i = \frac{\varepsilon_\mu{}^{ij} x_j}{er^2}(K-1), \qquad \varphi^i = \frac{x^i}{er^2} H$$

- spherically symmetric ansatz

$K(r),\, H(r)$ - unknown functions of radius $r$

$$r^2 K'' = K(K^2 + H^2 - 1)$$

$$r^2 H'' = 2HK^2 + \lambda\left(\frac{H^2}{e^2} - a^2 r^2\right)$$

- nonlinear system of equations

The Prasad-Sommerfield (1975) solution ( for $\lambda = 0$ ):

$$K = \frac{ear}{\mathrm{sh}(ear)}, \qquad H = \frac{ear}{\mathrm{th}(ear)} - 1$$

# Disclinations and dislocations

$\mathbb{SU}(2) \to \mathbb{SO}(3)$ acts simultaneously in coordinate and target spaces

Media without elastic stresses: $\mathbb{R}^3, \quad g_{\mu\nu} = \delta_{\mu\nu}, \quad e_\mu{}^i = \delta_\mu^i$

$$E = \frac{1}{4} F^{\mu\nu i} F_{\mu\nu i} + \frac{1}{2} \nabla^\mu \varphi^i \nabla_\mu \varphi_i + \frac{1}{4} \lambda (\varphi^2 - a^2)^2$$

- the free energy

$\varphi^i$ - sources for defects

$$\omega_\mu{}^{ij} = A_\mu{}^k \varepsilon_k{}^{ij} = (\delta_\mu^i x^j - \delta_\mu^j x^i) \frac{K-1}{er^2}$$

- spherically symmetric solution

$$R_{\mu\nu}{}^k = \varepsilon_{\mu\nu}{}^k \frac{K'}{er} - \frac{\varepsilon_{\mu\nu}{}^j x_j x^k}{er^3} \left( K' - \frac{K^2 - 1}{r} \right)$$

$$T_{\mu\nu}{}^k = (\delta_\mu^k x_\nu - \delta_\nu^k x_\mu) \frac{K-1}{er^2}$$

- continuous distribution of disclinations and dislocations

# Conclusion

1) There is a new $\mathbb{U}(1)$ gauge model without gauge field.

2) All ‘t Hooft-Polyakov type solutions have straightforward physical interpretation in the geometric theory of defects.

3) They describe continuous distribution of dislocations and disclinations in elastic media.