Discrete connection gravity and finite diagram technique

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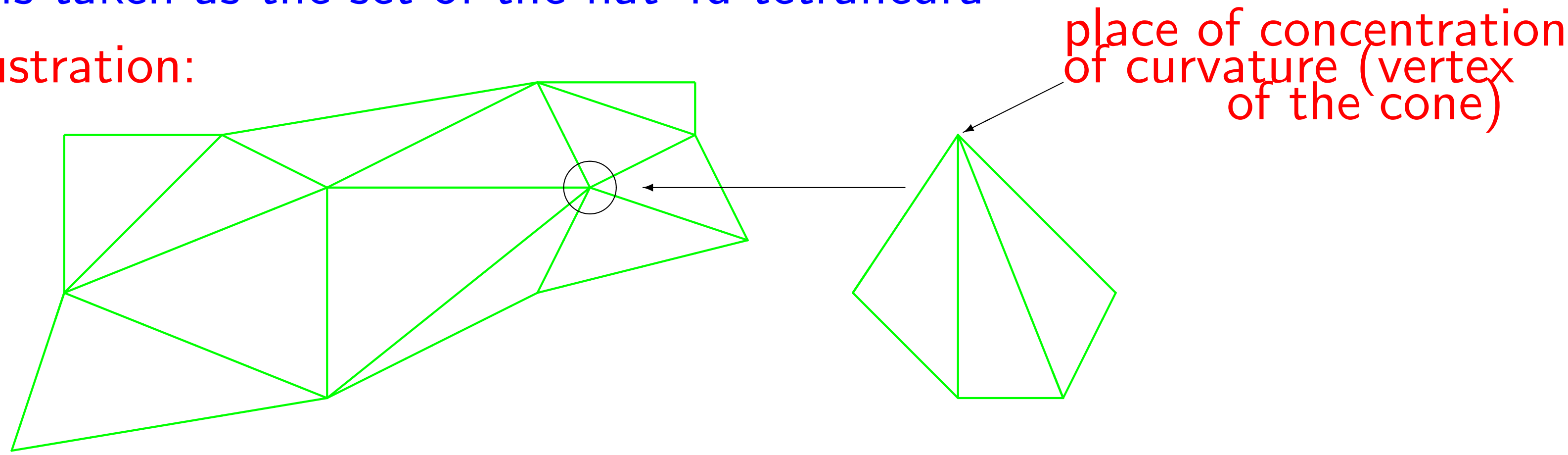
Piecewise flat spacetime

This class is sufficiently large to approximate any metric with any accuracy.

Spacetime is taken as the set of the flat 4d tetrahedra

$d = 2$ illustration:



σ^n is n-dim tetrahedron or n-simplex. Regge action

$$\frac{1}{2}\int R\sqrt{g}\,\mathrm{d}^4x = \sum_{\text{triangles}} (\text{area}) \cdot (\text{defect angle}) = \sum_{\sigma^2} A_{\sigma^2}\alpha_{\sigma^2}$$

This is expected to circumvent the nonrenormalizability problem of GR.

Discrete connection representation

$$S = \frac{1}{2}\int R\sqrt{-g}\,\mathrm{d}^4x = \frac{1}{2}\sum_{\sigma^2}\left[\left(1+\frac{i}{\gamma}\right)\sqrt{{}^+v^2_{\sigma^2}}\arcsin\frac{{}^+v_{\sigma^2} * {}^+R_{\sigma^2}(\Omega)}{\sqrt{{}^+v^2_{\sigma^2}}}\right.$$

$$\left. + \left(1-\frac{i}{\gamma}\right)\sqrt{{}^-v^2_{\sigma^2}}\arcsin\frac{{}^-v_{\sigma^2} * {}^-R_{\sigma^2}(\Omega)}{\sqrt{{}^-v^2_{\sigma^2}}}\right].$$

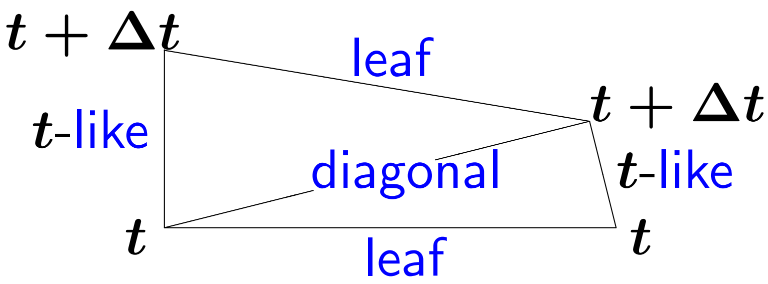
Here, $v * R \equiv \frac{1}{2}v^a R^{bc}\epsilon_{abc}$ ${}^\pm v_{ab} = {}^\pm v^k\, {}^\pm\Sigma_{kab}/2$ for $v^{ab} = \frac{1}{2}\epsilon^{ab}{}_{cd}l_1^c l_2^d$ we have $2\,{}^\pm\boldsymbol{v} = \pm i\boldsymbol{l}_1 \times \boldsymbol{l}_2 - \boldsymbol{l}_1 l_2^0 + \boldsymbol{l}_2 l_1^0$ selfdual-antiselfdual decomposition SO(3,1) $\subset$ SO(3,C) $\times$ SO(3,C) triple of (anti-)selfdual basis matrices ${}^\pm\Sigma_k$ obey algebra of the Pauli matrices times $-i$

$$R_{\sigma^2}(\Omega) = \prod_{\{\sigma^3:\ \sigma^3 \supset \sigma^2\}} \Omega_{\sigma^3}^{\epsilon(\sigma^2,\sigma^3)},$$ holonomy of Ω, $\epsilon(\sigma^2,\sigma^3) = \pm 1$

Integration over connection

$$S = \frac{1}{2}\left(1+\frac{i}{\gamma}\right)\Bigg[\sum_{\substack{\text{leaf/dia}-\\ \text{gonal } \boldsymbol{\sigma}^2}} \sqrt{v_{\sigma^2}^2}\,\mathbf{arcsin}\,\frac{v_{\sigma^2} * {}^{+}R_{\sigma^2}}{\sqrt{v_{\sigma^2}^2}}$$

$$+\sum_{t-\text{like } \boldsymbol{\sigma}^2} \sqrt{v_{\sigma^2}^2}\,\mathbf{arcsin}\,\frac{v_{\sigma^2} * {}^{+}R_{\sigma^2}(\{{}^{+}R_{\sigma^2}|_{\text{leaf/diagonal } \boldsymbol{\sigma}^2}\})}{\sqrt{v_{\sigma^2}^2}}\Bigg] + \begin{array}{c}\text{complex}\\ \text{conjugate}\end{array}$$



Here $v \equiv {}^{+}v$. $\int \exp(iS)\mathcal{D}\Omega$ is non-singular for τ_{σ^2} ($\equiv v_{\sigma^2}$ for t-like σ^2) tending to zero.

Factorization

$$\prod_{\sigma^3} \mathcal{D}\Omega_{\sigma^3} = \prod_{\mathrm{leaf/diagonal}\ \sigma^2} \mathcal{D}R_{\sigma^2} \prod_{\mathrm{leaf/diagonal}\ \sigma^3} \mathcal{D}\Omega_{\sigma^3}.$$

At $\tau_{\sigma^2} \to 0$ $\int \exp(iS)\mathcal{D}\Omega$ factorizes into product over leaf/diagonal triangles of integrals of the type

$$\int \exp i \left[\frac{1}{2}\left(1+\frac{i}{\gamma}\right)\sqrt{v^2}\,\mathrm{arcsin}\,\frac{v*{}^{+}R}{\sqrt{v^2}} + \text{complex conjugate}\right]\mathcal{D}R$$

For illustration, $\int \exp(i\mathrm{v}\ \mathrm{sh}\psi)\mathrm{d}\psi = \int \exp(-\mathrm{v}\ \mathrm{ch}\psi)\mathrm{d}\psi$ have exponential suppression like $\exp(-\mathrm{v})$. Here $\mathrm{v} = |\sqrt{v^2}|$ is area, ψ is Lorentzian rotational "angle". The typical integrals with actual "arcsin" turn out to be less singular at small v and stronger exponentially suppressed at large v. Those that arise in the following orders of expansion in τ_{σ^2} have the same behavior at small and large areas. In the leading term, this gives pure module of $\int \exp(iS)\mathcal{D}\Omega$.

Stationary phase expansion

In the leading term $\int \exp[i(S(v,\Omega_0)+(\omega B\omega)+O(\omega^3))]D\omega$
$=\exp[i(S(l)]\int\exp[i(\omega B\omega)](1+O(\omega^3))D\omega$ gives pure phase.
Here $\Omega=\Omega_0\exp\omega$, $\omega^{ab}=-\omega^{ba}$, and Ω_0 is a particular solution of the equations of motion for Ω so that $S(v,\Omega_0)$ is the Regge action $S(l)$, a function of the edge lengths l.
Compare with continuum $\det B$ singular at small lapse-shift. $\frac{1}{2}\int R\sqrt{-g}\mathrm{d}^4x$ is represented as (Palatini-Holst action)

$$\frac{1}{8}\int(\epsilon_{abcd}e^a_\lambda e^b_\mu+\frac{2}{\gamma}e_{\lambda c}e_{\mu d})\epsilon^{\lambda\mu\nu\rho}[\partial_\nu+\omega_\nu,\partial_\rho+\omega_\rho]^{cd}\mathrm{d}^4x$$

To estimate $\det B$, redefine variables as $\omega^{ab}_\lambda=(\det\|e^d_\mu\|)^{-1/2}e^c_\lambda w^{ab}_c$, then $(\omega B\omega)$ goes to $(wB'w)$, $B'=\mathrm{const.}$ $\det B$ is (the product over points of) a power of $\det\|e^a_\lambda\|\propto N$, singular at small N, a scale of Arnowitt-Deser-Misner lapse-shift e^a_0. $(\det B)^{-1/2}\propto\int\exp[i(\omega B\omega)]D\omega$ diverges at $N\to 0$ at large $\omega^{ab}_\alpha\propto N^{-1/2}$, $\alpha=1,2,3$ and small $\omega^{ab}_0\propto N^{1/2}$.

Quadratic form of connection

In continuum $\omega_\lambda = ({}^+\omega_\lambda \cdot {}^+\Sigma + {}^-\omega_\lambda \cdot {}^-\Sigma)/2$, $({}^-\omega)^* = {}^+\omega \equiv \omega$, and it is $\frac{1}{4}\left(1 + \frac{i}{\gamma}\right) \epsilon^{\alpha\beta\gamma} \left[\omega_\alpha \times \omega_\beta \cdot \left(ie_0 \times e_\gamma + e^0_0 e_\gamma\right) + \omega_0 \times \omega_\alpha \cdot (ie_\beta \times e_\gamma)\right] +$ compl. conj., where $e^0_\alpha = 0$ (Schwinger time gauge), $e^0_0 = N$, $e^a_0 = O(N)$. Divergence of $(\det B)^{-1/2} \propto \int \exp[i(\omega B \omega)] D\omega$ at $N \to 0$ comes from the region $\omega_\alpha \propto N^{-1/2}$, $\omega_0 \propto N^{1/2}$ while $(\omega B \omega)$ is finite.

In the discrete case, B is large, close to diagonal one, but is not diagonal. Effectively, more variables enter this expression, say, ω_λ and $\omega^\prime_\lambda$ from a neighboring point, and terms like $\omega^\prime_\alpha \times \omega_\alpha \cdot (ie_\beta \times e_\gamma)$ which are large $\propto N^{-1}$ in the "dangerous" region of integration $\omega^\prime_\alpha, \omega_\alpha \propto N^{-1/2}$ and suppress it through exponent.

This is in correspondence with the above mentioned non-singularity of the result of the functional integration over connection at small t-like edges and indicates that the stationary phase expansion is over the powers of l^{-2}, l being a scale of the *leaf/diagonal* lengths in this case.

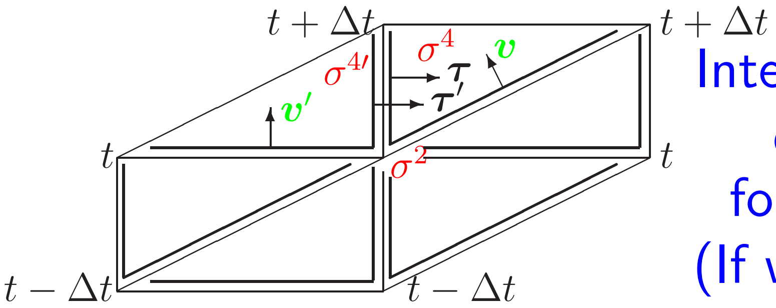
Starting point for perturbative expansion

Thus we have $\int F(l)\mathrm{d}^n l \exp(iS(l))$, $l = (l_1, \ldots, l_n)$, $F(l)$ has maxima. Introduce a new n-vector variable $u = (u_1, \ldots, u_n)$ to get Lebesgue measure $F(l)\mathrm{d}^n l = \mathrm{d}^n u$ and expand around $l_0 = l(u_0)$ in Taylor series $S(l(u)) =$ $S(l_0) + \sum_{j,k} \frac{\partial S(l_0)}{\partial l_j}\frac{\partial l_j(u_0)}{\partial u_k}\Delta u_k + \frac{1}{2}\sum_{j,k,l}\frac{\partial}{\partial u_l}\left(\frac{\partial l_j}{\partial u_k}\frac{\partial S}{\partial l_j}\right)\Big|_{u=u_0}\Delta u_k \Delta u_l + \ldots$ Einstein-Regge equations (absence of linear term) define $l_0 = (l_{01}, \ldots, l_{0n})$, but not completely. Take the simplest, flat solution. Let the spacetime be a hyperplane in an imaginary enveloping D-dimensional spacetime. Transformations $\delta_{||} l$ that leave vertices of Regge skeleton in the hyperplane do not change the action (zero), $\delta_\perp l$ that move vertices from a hyperplane change geometry. While $\delta_{||} S = 0 = \delta^2_{||} S$, generally $\delta_{||}\delta_\perp S$ and $\delta^2_\perp S$ are nonzero, and second variation matrix $\delta^2 S$ is nondegenerate. Det of Δu^2 form $F(l_0)^{-2} \det \left\| \frac{\partial^2 S(l_0)}{\partial l_i \partial l_k} \right\|$ is ∞ at the boundary points $F(l) \to 0$ (as we find, if any length is 0 or ∞) like an infinite potential wall in $S(l(u))$, and should be minimal at the optimal l_0.

Functional measure

Canonical measure $\mathrm{d}^6 v \mathcal{D}\Omega$ ($\mathcal{D}\Omega$ is Haar measure) follows in the continuum time limit, in which kinetic term is a combination of $\mathrm{tr}(v\Omega^\dagger \dot{\Omega})$ and $\mathrm{tr}({}^*v\Omega^\dagger \dot{\Omega})$ (${}^*v^{ab} \equiv \epsilon^{ab}{}_{cd} v^{cd}/2$). It can be symmetrically generalized to full discrete measure (which gives canonical one in the continuous time limit) as some product of $\mathrm{d}^6 v_{\sigma^2|\sigma^4} \mathcal{D}\Omega_{\sigma^3}$. To avoid singularity of discrete theories connected with symmetry changes, here at the flat background, we consider area tensors as independent variables (6 tensors $v_{\sigma^2|\sigma^4}$ in each 4-simplex σ^4). Then we project the measure in the configuration superspace onto the physical hypersurface where edge vectors leading to given area tensors exist by introducing δ-function factor $\int V^\eta \delta^{21}\left(\epsilon_{abcd} v^{ab}_{\lambda\mu} v^{cd}_{\nu\rho} - V\epsilon_{\lambda\mu\nu\rho}\right) \mathrm{d}V$ in each 4-simplex. It is general form of scalar density w. r. t. world indices (being scalar density is usual requirement for measure in metric gravity) proportional to such δ-function; at $\eta = 20$ it is scalar. One else δ-factor ensuring unambiguity of the induced metric on the 3-faces is fixed by requirement of invariance w. r. t. deformation of 3-faces in their planes.

Background area/length scale



Integrations concerning given triangle σ^2 enter full discrete measure in the form $\mathrm{d}^6 v_{\sigma^2|\sigma^4} \mathrm{d}^6 v_{\sigma^2|\sigma^{4\prime}}$ for two containing it "future" 4-simplices $\sigma^4, \sigma^{4\prime}$ (If we assign (six) area tensors to the earlier vertex of the t-like edge in each 4-simplex.) On physical hypersurface, area scale $\mathrm{v}_{\sigma^2|\sigma^4} = \mathrm{v}_{\sigma^2|\sigma^{4\prime}} \equiv \mathrm{v}$, single out v-part: $\mathrm{d}^6 v_{\sigma^2|\sigma^4} \mathrm{d}^6 v_{\sigma^2|\sigma^{4\prime}} = \mathrm{v}^{11} \mathrm{d}\mathrm{v} \ldots$. $\mathcal{D}\Omega$ part of the measure upon integrating over it gives certain $\mathcal{N}(\mathrm{v}, \mathrm{v}^*)$ so that in overall $\mathcal{N} \mathrm{v}^{11} \mathrm{d}\mathrm{v} = \left| \frac{1}{\frac{1}{4}\left(\frac{1}{\gamma} - i\right)^2 \mathrm{v}^2 + 1} \frac{\frac{1}{4}\left(\frac{1}{\gamma} - i\right)\mathrm{v}}{\mathrm{sh}\left[\frac{\pi}{2}\left(\frac{1}{\gamma} - i\right)\mathrm{v}\right]} \right|^2 \mathrm{v}^{11} \mathrm{d}\mathrm{v}$. Neighboring triangles are not independent, area scale in a group of T such triangles being v, the measure is $(\mathcal{N} \mathrm{v}^{12})^T \mathrm{v}^{-1} \mathrm{d}\mathrm{v}$. As considered above, taking into account linear dependence of S on v type variables, we should maximize $(\mathcal{N} \mathrm{v}^{12})^T \mathrm{v}^{-1/2}$, or, at large T, $\mathcal{N} \mathrm{v}^{12}$. We estimate optimal $2\mathrm{v}_0 = ia^2$, $a \simeq \sqrt{24/\pi}$ or, for general η, $\sqrt{2(\eta - 8)/\pi}$.

Form of propagator

In the above measure, 4 conditions per vertex should be imposed on the non-dynamical t-like tensors to lead to usual gauge fixing on large scale. Since the leaf/diagonal edge lengths are (loosely) fixed dynamically, these 4 conditions remain just to fix the discrete lapse-shift vectors. Particular case of such a gauge is the synchronous frame gauge ($l^0_0 = 1, l_0 = 0$). The discrete propagator is a simplicial analog of the continuum graviton propagator in the synchronous frame gauge $\mathrm{d}s^2 = \gamma_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta - \mathrm{d}t^2$, $G_{\alpha\beta}{}^{\gamma\delta}\sqrt{-g} = 2\frac{L^\gamma_\alpha L^\delta_\beta + L^\delta_\alpha L^\gamma_\beta - L_{\alpha\beta}L^{\gamma\delta}}{p^2_0 - p^2}$, $L_{\alpha\beta} = \gamma_{\alpha\beta} - \frac{p_\alpha p_\beta}{p^2_0}$ (if the background $\gamma_{\alpha\beta} = const$). Knowing propagator as functional of Regge lattice is unreal, but for estimate we can use some finite difference or cubic lattice form at the known scale a, $a^2 G_{\alpha\beta}{}^{\gamma\delta} = 2\frac{\mathcal{L}^\gamma_\alpha \mathcal{L}^\delta_\beta + \mathcal{L}^\delta_\alpha \mathcal{L}^\gamma_\beta - \mathcal{L}_{\alpha\beta}\mathcal{L}^{\gamma\delta}}{\sin^2 p_0 - \sum_\alpha \sin^2 p_\alpha}$, $\mathcal{L}_{\alpha\beta} = \gamma_{\alpha\beta} - \frac{\sin p_\alpha \sin p_\beta}{\sin^2 p_0}$. Here $a^2 G_{\alpha\beta}{}^{\gamma\delta}$ has a sense of the length variation correlator $\langle \Delta l_j \Delta l_k \rangle$.

Conclusions

1) The result of the functional integration over connection can be written as $(F + {}^{(1)}F + {}^{(2)}F + \dots)\exp[i(S + {}^{(1)}S + {}^{(2)}S + \dots)]$, expansion over lapse-shift is taken for module, stationary phase expansion for phase is in powers of inverse length scale squared l^{-2} of leaf/diagonal (at small t-like ones) edges. Later l^{-2} is replaced by a^{-2}, which can serve a small parameter at large η.

2) The perturbative expansion goes around a point which is partially fixed by eqs of motion in Taylor expansion of action, but some degrees of freedom like length scale which are not fixed in this way are fixed in a finer manner by minimizing Det of 2nd order form (in variables in which measure is Lebesgue).

3) The discrete propagator is an analog of the continuum graviton propagator in *the synchronous frame gauge* on Regge lattices appropriately fixed at areas a^2. For estimate we can take its cubic lattice form with the spacings a.

4) Non-perturbative nature of the propagator shows up in the dependence on $a \simeq \sqrt{2(\eta - 8)/\pi}$, where η specifies measure (volume factor $V_{\sigma^4}^{\Delta\eta}$ there).

Thank you for attention!