$\mathcal{N}$-extended supersymmetric Calogero models

Sergey Krivonos

Bogoliubov Laboratory of Theoretical Physics
Joint Institute for Nuclear Research
Dubna, Russia

In collaboration with O. Lechtenfeld and A. Sutulin

Main result

We propose a new $\mathcal{N}$-extended supersymmetric $su(n)$ spin-Calogero model. Employing a generalized Hamiltonian reduction adopted to the supersymmetric case, we explicitly construct a novel rational n-particle Calogero model with an arbitrary even number of supersymmetries. It features $\mathcal{N}n^2$ rather than $\mathcal{N}n$ fermionic coordinates and increasingly high fermionic powers in the supercharges and the Hamiltonian.

Plan

- Introduction
- Bosonic Calogero model from hermitian matrices
- $\mathcal{N}$-extended supersymmetric $su(n)$ spin-Calogero model
- Comments on the calculations
- $\mathcal{N}$-extended supersymmetric (no-spin) Calogero models
- Simplest example: $\mathcal{N}=2$ supersymmetric two-particle Calogero model
- Conclusion

The $N = 4$ supersymmetrization of the simplest n-particles model describing the system of identical particles scattering on the line with inverse-square interaction potentials, as first introduced by F. Calogero being the puzzle for a long time. Despite the simplicity of its hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{2}\sum_{i\neq j} \frac{g^2}{(x^i - x^j)^2},$$

the all attempts to construct its $N = 4$ supersymmetric version beyond the four-particles case were unsuccessful. In the same time the $N = 2$ supersymmetric Calogero model has been constructed many years ago by D.Z. Freedman and P.F. Mende.

- The first attempt to construct the N = 4 supersymmetric extension of the Calogero models was performed by N. Wyllard with the completely discouraging result: it was shown that such a system does not exist at all;
- The next important step has been done by A. Galajinsky, O. Lechtenfeld and K. Polovnikov who explicitly constructed the supercharges and Hamiltonian for the $N = 4$ supersymmetric three-particles Calogero model. In his paper they showed that the terms which spoil the construction by Wyllard can be interpreted as quantum corrections. So, in the classical limit the proper action could be obtained. Unfortunately, the component description in the Hamiltonian formalism presented in his paper being extended to $n > 3$ particles cases leads to a very complicated systems of equations for which even the proof of existence of solutions is rather nontrivial;
- The superspace description of the $N = 4$ supersymmetric three-particles Calogero model has been constructed by S. Bellucci, S. Krivonos and A. Sutulin;
- Finally, the four-particles $N = 4$ supersymmetric Calogero model has been constructed within the Hamiltonian formalism by S. Krivonos and O. Lechtenfeld.

It seems that a guiding principle was missing for the construction of extended supersymmetric Calogero models. Indeed, while for $n \leq 3$ translation and (super-)conformal symmetry almost completely defines the system, the $n \geq 4$ cases admit a lot of freedom which cannot *a priori* be fixed. In the bosonic case, such a guiding principle exists. The Calogero model as well as its different extensions are closely related with matrix models and can be obtained from them by a reduction procedure (the first results in this direction have been obtained by A. Polychronakos). If we want to employ this principle also for finding extended supersymmetric Calogero models, then the two main steps are

- supersymmetrization of a matrix model
- supersymmetrization of the reduction procedure or proper gauge fixing.

This idea is not new. It has successfully been employed in the series of papers of S. Fedoruk, E. Ivanov, O. Lechtenfeld and S. Sidorov.

The resulting supersymmetric systems feature

- a large number of fermions – far more than the $4n$ fermions expected in an N$=4$ n-particle system within the standard (but unsuccessful!) approach
- a rather complicated structure of the supercharges and the Hamiltonian, with fermionic polynomials of maximal degree
- a variety of bosonic potentials, including $su(2)$ *spin*-Calogero interactions

but they do not contain a genuine N$=4$ supersymmetric Calogero model, i.e. one with a mere pairwise inverse-square no-spin bosonic potential.

Here we use the same guiding principle and start with the bosonic $su(n)$ spin-Calogero model in the Hamiltonian approach. We then provide an $\mathcal{N}$-extended supersymmetrization of this system. It is important that we do *not a priori* fix a realization for the $su(n)$ generators. Finally we generalize the reduction procedure to the $\mathcal{N}$-extended system and find the first $\mathcal{N}$-extended supersymmetric Calogero model, for *any* even number of supersymmetries.

It is well known that the rational n-particle Calogero model can be obtained by Hamiltonian reduction from the hermitian matrix model. Adapted to our purposes, the procedure reads as follows. One starts from the $su(n)$ spin generalization of the standard Calogero model, as given by

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{2}\sum_{i\neq j}^{n} \frac{\ell_{ij}\ell_{ji}}{(x^i - x^j)^2} \,.$$

The particles are described by their coordinates x^i and momenta p_i together with their internal degrees of freedom encoded in the angular momenta $(\ell_{ij})^\dagger = \ell_{ji}$ with $\sum_i \ell_{ii} = 0$. The non-vanishing Poisson brackets are

$$\left\{x^i, p_j\right\} = \delta^i_j \qquad \text{and} \qquad \{\ell_{ij}, \ell_{km}\} = \mathrm{i}\left(\delta_{im}\ell_{kj} - \delta_{kj}\ell_{im}\right) \,.$$

To get the standard Calogero Hamiltonian one has to reduce the angular sector of the latter, in two steps. Firstly, one (weakly) imposes the constraints

$$\ell_{11} \approx \ell_{22} \approx \ldots \approx \ell_{nn} \approx 0 \, .$$

They commute with the Hamiltonian and with each other, hence are of first class. To resolve them one introduces auxiliary complex variables v_i and $\bar{v}_i = (v_i)^\dagger$ obeying the Poisson brackets

$$\{v_i, \bar{v}_j\} = -\mathrm{i}\, \delta_{ij}$$

and realizes the $su(n)$ generators ℓ_{ij} as

$$\hat{\ell}_{ij} = -v_i \bar{v}_j + \frac{1}{n} \delta_{ij} \sum_{k}^{n} v_k \bar{v}_k \, .$$

Secondly, passing to polar variables r_i and ϕ_i defined as

$$v_i = r_i e^{\mathrm{i}\phi_i} \qquad \text{and} \qquad \bar{v}_i = r_i e^{-\mathrm{i}\phi_i} \qquad \Rightarrow \qquad \{r_i, \phi_j\} = \frac{1}{2r_i} \delta_{ij} \, ,$$

the constraints are resolved by putting

$$r_1 \approx r_2 \approx \ldots \approx r_n \, .$$

Plugging this solution into the Hamiltonian one may additionally fix $n-1$ angles ϕ_i, say

$$\phi_1 \approx \phi_2 \approx \ldots \approx \phi_{n-1} \approx 0 \, .$$

At this stage the $2n$ variables $\{r_i, \phi_i\}$ are reduced to the two variables r_n and ϕ_n. However, the reduced Hamiltonian does not depend on ϕ_n and has the form

$$H_{\rm red} = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{2}\sum_{i\neq j}^{n} \frac{r_n^4}{(x^i - x^j)^2} \ .$$

Therefore

$$\{H_{\rm red}, r_n\} \approx 0 \qquad \text{and} \qquad r_n^2 \approx \text{const} =: g \ ,$$

and the reduced Hamiltonian $H_{\rm red}$ coincides with the standard n-particle rational Calogero Hamiltonian.

In what follows we will construct an $\mathcal{N}$-extended supersymmetric generalization of the $su(n)$ spin Calogero model Hamiltonian and perform the supersymmetric version of the reduction just discussed, finishing with an $\mathcal{N}$-extended supersymmetric Calogero model, for $\mathcal{N} = 2M$ and $M = 1, 2, 3, \ldots$.

First step is to clarify what is the minimal number of fermionic variables necessary to realize an $\mathcal{N}=2M$ supersymmetric extension of the $su(n)$ spin-Calogero model. Clearly, as partners to the bosonic coordinates x^i one needs $\mathcal{N}n$ fermions ψ_i^a and $\bar\psi_{i\,a}$ with $a=1,2,\ldots M$. However, this is not enough to construct $\mathcal{N}$ supercharges Q^a and $\overline{Q}_b$ which must generate the $\mathcal{N}=2M$ superalgebra

$$\left\{Q^a,\overline{Q}_b\right\}=-2\mathrm{i}\,\delta^a_b\,H \qquad \text{and} \qquad \left\{Q^a,Q^b\right\}=\left\{\overline{Q}_a,\overline{Q}_b\right\}=0\,.$$

The reason is simple: to generate the potential term $\sum_{i\neq j}^n \frac{\ell_{ij}\ell_{ji}}{\left(x^i-x^j\right)^2}$ in the Hamiltonian, the supercharges Q^a and $\overline{Q}_b$ must contain the terms

$$\mathrm{i}\sum_{i\neq j}^n\frac{\ell_{ij}\,\rho^a_{ji}}{x^i-x^j} \qquad \text{and} \qquad -\,\mathrm{i}\sum_{i\neq j}^n\frac{\ell_{ji}\,\bar\rho_{ij\,b}}{x^i-x^j}\,,$$

respectively, where ρ^a_{ij} and $\bar\rho_{ij\,a}$ are some additional fermionic variables. These fermions cannot be constructed from ψ_i^a or $\bar\psi_{i\,a}$. Hence, we are forced to introduce $\mathcal{N}n(n-1)$ further independent fermions ρ^a_{ij} and $\bar\rho_{ij\,a}$ subject to $\rho^a_{ii}=\bar\rho_{ii\,a}=0$ for each value of the index i.

In total, we thus utilize $\mathcal{N}n^2$ fermions of type ψ or ρ, which we demand to obey the following Poisson brackets,

$$\left\{\psi_i^a, \bar{\psi}_{j\,b}\right\} = -\mathrm{i}\,\delta_b^a\delta_{ij}\ ,\ \left\{\rho_{ij}^a, \bar{\rho}_{km\,b}\right\} = -\mathrm{i}\,\delta_b^a\delta_{im}\delta_{jk}\ ,\ \text{with}\ \left(\rho_{ij}^a\right)^\dagger = \bar{\rho}_{ji\,a}\ \text{and}\ \rho_{ii}^a = \bar{\rho}_{ii\,a} = 0\ .$$

The next important ingredient of our construction is the composite object

$$\Pi_{ij} = \sum_{a=1}^{M}\Big[\left(\psi_i^a - \psi_j^a\right)\bar{\rho}_{ij\,a} + \left(\bar{\psi}_{i\,a} - \bar{\psi}_{j\,a}\right)\rho_{ij}^a + \sum_{k=1}^{n}\left(\rho_{ik}^a\bar{\rho}_{kj\,a} + \bar{\rho}_{ik\,a}\rho_{kj}^a\right)\Big] \qquad \Rightarrow \qquad \left(\Pi_{ij}\right)^\dagger = \Pi_{ji}\ .$$

One may check that the Π_{ij} form an $su(n)$ algebra just like the ℓ_{ij},

$$\left\{\Pi_{ij}, \Pi_{km}\right\} = \mathrm{i}\left(\delta_{im}\Pi_{kj} - \delta_{kj}\Pi_{im}\right)\ .$$

It is a matter of straightforward calculation to check that the supercharges

$$Q^a = \sum_{i=1}^{n} p_i \psi_i^a + \mathrm{i} \sum_{i \neq j}^{n} \frac{(\ell_{ij} + \Pi_{ij})\, \rho_{ji}^a}{x^i - x^j} \qquad \text{and} \qquad \overline{Q}_b = \sum_{i=1}^{n} p_i \bar{\psi}_{i\,b} - \mathrm{i} \sum_{i \neq j}^{n} \frac{\bar{\rho}_{ij\,b}\,(\ell_{ji} + \Pi_{ji})}{x^i - x^j}$$

obey the $\mathcal{N}=2M$ superalgebra with the Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \frac{1}{2} \sum_{i \neq j}^{n} \frac{(\ell_{ij} + \Pi_{ij})\,(\ell_{ji} + \Pi_{ji})}{(x^i - x^j)^2} \,,$$

modulo the first-class constraints

$$\chi_i := \ell_{ii} + \Pi_{ii} \approx 0 \quad \forall\, i \,,$$

with

$$\{Q^a, \chi_i\} \approx \left\{\overline{Q}_a, \chi_i\right\} \approx \{H, \chi_i\} \approx \{\chi_i, \chi_j\} \approx 0 \,.$$

These supercharges Q^a and $\overline{Q}_b$ and the Hamiltonian H describe the $\mathcal{N}=2M$ supersymmetric $su(n)$ spin-Calogero model.

For N$=4$ it essentially coincides with the $osp(4|2)$ supersymmetric mechanics constructed in S. Fedoruk, E. Ivanov, O. Lechtenfeld, Phys. Rev. D **79** (2009) 105015, S. Fedoruk, E. Ivanov, O. Lechtenfeld, S. Sidorov, `arXiv:1801.00206[hep-th]`. However, there are a few differences:

- Our Hamiltonian has no interaction for the center-of-mass coordinate $X = \sum_i x^i$. Correspondingly, the supercharges do not include certain terms which appeared in above papers.
- Working at the Hamiltonian level, we may keep the $su(n)$ generators ℓ_{ij} unspecified. Precisely this enables the minimal realization with a minimal number of auxiliary variables $v_i, \bar{v}_i$. At the Lagrangian level this corresponds to using $(2,4,2)$ supermultiplets for the auxiliary bosonic superfields instead of $(4,4,0)$ superfields as in the cited papers.

One may wonder how to perform the above calculations for arbitrary number of particles and arbitrary number of supersymmetries? The simplest way to perform these calculations is to introduce the composite object $L_{ij} \equiv \ell_{ij} + \Pi_{ij}$. The L_{ij} still form $su(n)$ algebra like ℓ_{ij} and Π_{ij}

$$\{L_{ij}, L_{km}\} = \mathrm{i}\left(\delta_{im} L_{kj} - \delta_{kj} L_{im}\right) \ ,$$

and they commute with the fermions exactly as

$$\left\{L_{ij}, \psi^a_k\right\} = \mathrm{i}\left(\delta_{ik} - \delta_{jk}\right) \rho^a_{ij} \ , \qquad \left\{L_{ij}, \rho^a_{km}\right\} = -\mathrm{i}\,\delta_{im}\delta_{jk}\left(\psi^a_i - \psi^a_j\right) - \mathrm{i}\delta_{jk}\,\rho^a_{im} + \mathrm{i}\delta_{im}\,\rho^a_{kj} \ ,$$

$$\left\{L_{ij}, \bar{\psi}_{k\,a}\right\} = \mathrm{i}\left(\delta_{ik} - \delta_{jk}\right) \bar{\rho}_{ij\,a} \ , \quad \left\{L_{ij}, \bar{\rho}_{km\,a}\right\} = -\mathrm{i}\,\delta_{im}\delta_{jk}\left(\bar{\psi}_{i\,a} - \bar{\psi}_{j\,a}\right) - \mathrm{i}\delta_{jk}\,\bar{\rho}_{im\,a} + \mathrm{i}\delta_{im}\bar{\rho}_{kj\,a} \ .$$

It should be clear now, that the closing of the superalgebra for the supercharges

$$Q^a = \sum_{i=1}^{n} p_i \psi^a_i + \mathrm{i} \sum_{i \neq j}^{n} \frac{L_{ij}\, \rho^a_{ji}}{x^i - x^j} \qquad \text{and} \qquad \overline{Q}_b = \sum_{i=1}^{n} p_i \bar{\psi}_{i\,b} - \mathrm{i} \sum_{i \neq j}^{n} \frac{\bar{\rho}_{ij\,b}\, L_{ji}}{x^i - x^j}$$

does not depend on the number of particles and number of supersymmetries and it is based on the above commutation relations only.

The unique additional important equality widely used in the calculations is

$$\frac{1}{(x^i - x^j)(x^i - x^k)} + \frac{1}{(x^j - x^i)(x^j - x^k)} + \frac{1}{(x^k - x^i)(x^k - x^j)} = 0 \quad \text{for} \quad i \neq j \neq k.$$

As we can see above, the supersymmetric analogs $\ell_{ii} + \Pi_{ii} \approx 0$ of the purely bosonic constraints $\ell_{ii} \approx 0$ appear automatically. These constraints generate $n-1$ local U(1) transformations of the variables $\{v_i, \bar{v}_i, \rho_{ij}^a, \bar{\rho}_{ij\,a}\}$. In terms of the $2n$ polar variables r_i and ϕ_i defined previously, these constraints can be easily resolved as

$$r_k^2 \approx r_n^2 + \Pi_{kk} - \Pi_{nn} \qquad \text{for} \quad k = 1, \ldots, n-1 \; .$$

After fixing the residual gauge freedom as

$$\phi_1 \approx \phi_2 \approx \ldots \approx \phi_{n-1} \approx 0 \; ,$$

we obtain the supercharges and Hamiltonian which still obey the $\mathcal{N} = 2M$ superalgebra and contain only the surviving pair (r_n, ϕ_n) of the originally $2n$ “angular” variables. One may check that the supercharges Q^a and $\overline{Q}_b$ and the Hamiltonian H, with the generators ℓ_{ij} replaced by $\hat{\ell}_{ij}$ and with the above partial solution of the constraints taken into account, perfectly commute with $r_n^2 - \Pi_{nn}$. Thus, the final step of the reduction is to impose the constraint

$$r_n^2 - \Pi_{nn} \approx \text{const} =: g$$

and to fix the remaining U(1) gauge symmetry via

$$\phi_n \approx 0 \; .$$

The previous two relations are the supersymmetric analogs of $r_n^2 \approx \text{const} =: g$. We conclude that the full set of the reduction constraints reads

$$r_i^2 \approx g + \Pi_{ii} \qquad \text{and} \qquad \phi_i \approx 0 \qquad \text{for} \quad i = 1, \ldots, n \; .$$

With these constraints taken into account, our supercharges Q^a and $\overline{Q}_b$ and the Hamiltonian H acquire the form

$$\widehat{Q}^a = \sum_{i=1}^n p_i \psi_i^a - \mathrm{i} \sum_{i \neq j}^n \frac{\left(\sqrt{g+\Pi_{ii}}\sqrt{g+\Pi_{jj}} - \Pi_{ij}\right) \rho_{ji}^a}{x^i - x^j} ,$$

$$\widehat{\overline{Q}}_b = \sum_{i=1}^n p_i \bar{\psi}_{i\,b} + \mathrm{i} \sum_{i \neq j}^n \frac{\bar{\rho}_{ij\,b}\left(\sqrt{g+\Pi_{ii}}\sqrt{g+\Pi_{jj}} - \Pi_{ji}\right)}{x^i - x^j} ,$$

$$\widehat{H} = \frac{1}{2}\sum_{i=1}^n p_i^2 + \frac{1}{2}\sum_{i \neq j}^n \frac{\left(\sqrt{g+\Pi_{ii}}\sqrt{g+\Pi_{jj}} - \Pi_{ij}\right)\left(\sqrt{g+\Pi_{ii}}\sqrt{g+\Pi_{jj}} - \Pi_{ji}\right)}{\left(x^i - x^j\right)^2} .$$

It is matter of quite lengthy and tedious calculations to check that these supercharges and Hamiltonian form an $\mathcal{N}=2M$ superalgebra. The main complication arises from the expressions $\sqrt{g+\Pi_{ii}}$ present in the supercharges and the Hamiltonian. Due to the nilpotent nature of Π_{ij}, the series expansion eventually terminates, but even in the two-particle case with $N=4$ supersymmetry we encounter a lengthy expression,

$$\sqrt{g+\Pi_{11}} = \sqrt{g}\left(1 + \tfrac{1}{2g}\Pi_{11} - \tfrac{1}{8g^2}\Pi_{11}^2 + \tfrac{1}{16g^3}\Pi_{11}^3 - \tfrac{5}{128g^4}\Pi_{11}^4\right) .$$

For n particles the series will end with a term proportional to $(\Pi_{ii})^{\mathcal{N}(n-1)}$. Clearly, these terms will generate higher-degree monomials in the fermions, both for the supercharges and for the Hamiltonian.

For $\mathcal{N}=2$ supersymmetry one has to put $M=1$ in the expressions for the supercharges and Hamiltonian. This somewhat reduces their complexity compared to the N$=4$ case, but the real simplification occurs for two particles. Indeed, for $n=2$ we get

$$\Pi_{22}=-\Pi_{11} \qquad \text{and} \qquad \Pi_{11}^3\equiv 0 \;\Rightarrow\; \sqrt{g+\Pi_{11}}\sqrt{g-\Pi_{11}}=\left(g-\tfrac{1}{2g}\Pi_{11}^2\right) \text{ for } g\neq 0\,.$$

Moreover, the term Π_{11}^2 is of the maximal possible power in the ρ and $\bar\rho$ fermions and, therefore, disappears from the supercharges. Thus, we are left with

$$\widehat{Q}_{(2)}=\sum_{i=1}^{2}p_i\psi_i-\mathrm{i}\sum_{i\neq j}^{2}\frac{(g-\Pi_{ij})\,\rho_{ji}}{x^i-x^j} \qquad \text{and} \qquad \widehat{\bar{Q}}_{(2)}=\sum_{i=1}^{2}p_i\bar\psi_i+\mathrm{i}\sum_{i\neq j}^{2}\frac{\bar\rho_{ij}\,(g-\Pi_{ji})}{x^i-x^j}\,,$$

which have the standard structure – linear and cubic in the fermions. The Hamiltonian $\widehat{H}_{(2)}$ reduces to

$$\widehat{H}_{(2)}=\frac{1}{2}\sum_{i=1}^{2}p_i^2+\frac{g^2-\Pi_{11}^2-g\,(\Pi_{12}+\Pi_{21})+\Pi_{12}\Pi_{21}}{(x^1-x^2)^2}\,,$$

with the explicit expressions $\Pi_{11}=\rho_{12}\bar\rho_{21}+\bar\rho_{12}\rho_{21}$ and

$$\Pi_{12}=(\psi_1-\psi_2)\,\bar\rho_{12}+\left(\bar\psi_1-\bar\psi_2\right)\rho_{12}\,,\;\; \Pi_{21}=(\psi_2-\psi_1)\,\bar\rho_{21}+\left(\bar\psi_2-\bar\psi_1\right)\rho_{21}\,.$$

We propose a novel $\mathcal{N}$-extended supersymmetric $su(n)$ spin-Calogero model as a direct supersymmetrization of the bosonic $su(n)$ model. In the case of $N=4$ supersymmetry, our model resembles the one constructed in S. Fedoruk, E. Ivanov, O. Lechtenfeld, Phys. Rev. D **79** (2009) 105015, S. Fedoruk, E. Ivanov, O. Lechtenfeld, S. Sidorov, `arXiv:1801.00206[hep-th]`. However, there are two main differences:

- the center of mass is free
- the $su(n)$ generators are not specified in a particular realization.

Thanks to these features, we were able to generalize the reduction procedure to the no-spin Calogero model from $N=4$ supersymmetry to any number $\mathcal{N}=2M$ of supersymmetries. This lead to the discovery of a genuine $\mathcal{N}=2M$ supersymmetric rational Calogero model for any number of particles.
Our models belong to same class which was proposed in above cited papers. Its main features are

- a huge number of fermionic coordinates, namely $\mathcal{N}n^2$ in number rather than the $\mathcal{N}n$ to be expected
- the supercharges and the Hamiltonian contain terms which a fermionic power much larger than three.

Clearly, these features merit a more careful and detailed analysis.

The following further developments come to mind:

- a superspace description of the constructed models, at least for $\mathcal{N}=2$ and $\mathrm{N}=4$ supersymmetry, presumably with nonlinear chiral supermultiplets
- an extension to the Calogero–Sutherland inverse-sine-square model
- an extension to the Euler–Calogero–Moser system and its reduction to the goldfish system, yielding a supersymmetric goldfish model upon reduction (Anton Sutulin talk).