BRST approach to Lagrangian construction for bosonic continuous spin field

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Outlook

1. Spin-tensor representation for continuous spin field equations.

2. Lagrangian construction for bosonic continuous spin field with the help of BRST method

The talk is base on arXiv:1806.01640 [hep-th].

Continuous spin equations

Continuous spin representation is characterized by the following eigenvalues of the Casimir operators

$$P^2\Psi = 0 \qquad\qquad W^2\Psi = \mu^2\Psi$$

In order to obtain the equations in explicit form we need the explicit expressions for the operators entering into the Casimir operators.

Such expressions for the operators depend on how Ψ transforms under action of the Poincaré group.

Continuous spin equations

The Wigner's equations

Let us introduce an auxiliary 4-dimensional vector w^μ and define

$$\varphi_s(x,w) \quad = \quad \varphi_{\mu_1\ldots\mu_s}(x)\, w^{\mu_1}\ldots w^{\mu_s}$$

where $\varphi_{\mu_1\ldots\mu_s}(x)$ is a totally symmetric tensor field.

Then the spin momentum operator for $\varphi_s(x,w)$ is realized as follows

$$M_{\mu\nu} = w_\mu i \frac{\partial}{\partial w^\nu} - w_\nu i \frac{\partial}{\partial w^\mu}.$$

Continuous spin equations

The Wigner's equations

Using this expression for the spin momentum operator one can obtain the Wigner's equations
[V. Bargmann and E.P. Wigner, Proc.Nat.Acad.Sci. **34** (1948) 211]

$$p^2\Psi(p,w) = 0 \qquad (p_\nu w^\nu + \mu)\Psi(p,w) = 0$$

$$p^\nu \frac{\partial}{\partial w^\nu}\Psi(p,w) = 0 \qquad \left(\frac{\partial}{\partial w^\nu}\frac{\partial}{\partial w_\nu} + 1\right)\Psi(p,w) = 0$$

If these four equations are satisfied, then the field Ψ will satisfy

$$P^2\Psi = 0 \qquad W^2\Psi = \mu^2\Psi$$

and thus will describe the irreducible representation of the Poincare group with continuous spin.

Continuous spin equations
Spin-tensor representation

There is another possibility to realize the spin momentum operator

One can consider the spin-tensor field $\varphi_{a_1 \ldots a_n \dot{b}_1 \ldots \dot{b}_k}(x)$ with n undotted and k dotted indices and define

$$\varphi_{n,k}(x, \xi, \bar{\xi}) \;=\; \varphi_{a_1 \ldots a_n \dot{b}_1 \ldots \dot{b}_k}(x)\, \xi^{a_1} \ldots \xi^{a_n}\, \bar{\xi}^{\dot{b}_1} \ldots \bar{\xi}^{\dot{b}_k},$$

where we have introduced two auxiliary bosonic 2-dimensional spinors ξ^a and $\xi^{\dot{b}}$ of the Lorentz group.

In this representation the spin momentum operator looks like

$$M_{\mu\nu} \;=\; \sigma^{ab}_{\mu\nu} M_{ab} - \bar{\sigma}^{\dot{a}\dot{b}}_{\mu\nu} M_{\dot{a}\dot{b}}$$

$$M_{ab} \;=\; -\frac{i}{2}\Big(\xi_a \pi_b + \xi_b \pi_a\Big) \qquad \pi_a = -i\frac{\partial}{\partial \xi^a}$$

$$\bar{M}_{\dot{a}\dot{b}} \;=\; -\frac{i}{2}\Big(\bar{\xi}_{\dot{a}} \bar{\pi}_{\dot{b}} + \bar{\xi}_{\dot{b}} \bar{\pi}_{\dot{a}}\Big) \qquad \bar{\pi}_{\dot{a}} = -i\frac{\partial}{\partial \bar{\xi}^{\dot{a}}}$$

Continuous spin equations
Spin-tensor representation

Due to the identity

$$(\bar{\xi}\bar{\sigma}^{\mu}\pi)(\bar{\pi}\bar{\sigma}^{\nu}\xi)P_{\mu}P_{\nu} = (\xi\sigma^{\mu}\bar{\xi})(\bar{\pi}\bar{\sigma}^{\nu}\pi)P_{\mu}P_{\nu} + \bar{\xi}^{\dot{a}}\bar{\pi}_{\dot{a}}(i + \pi_a\xi^a)P^2$$

there are several equivalent forms for the second Casimir operator W^2 in the spin-tensor representation. One of them is

$$\begin{aligned} W^2 &= (\xi\sigma^{\mu}\bar{\xi})(\bar{\pi}\bar{\sigma}^{\nu}\pi)P_{\mu}P_{\nu} \\ &\quad + \frac{1}{2}\Big(M_{ab}M^{ab} + \bar{M}_{\dot{a}\dot{b}}\bar{M}^{\dot{a}\dot{b}} + \bar{\xi}^{\dot{a}}\bar{\pi}_{\dot{a}}\xi^a\pi_a\Big)P^2 \end{aligned}$$

Continuous spin equations

Spin-tensor representation

Using this expression for the second Casimir operator one can obtain equations for continuous spin field in spin-tensor representation

$$
\begin{aligned}
& p^2\Psi(p,\xi,\bar{\xi}) = 0, \\
& ((\bar{\pi}\bar{\sigma}^{\nu}\pi)p_{\nu} + i\mu)\Psi(p,\xi,\bar{\xi}) = 0, \\
& ((\xi\sigma^{\mu}\bar{\xi})p_{\mu} - i\mu)\Psi(p,\xi,\bar{\xi}) = 0.
\end{aligned}
$$

If these three equations are satisfied, then the field $\Psi(p,\xi,\bar{\xi})$ will satisfy

$$P^2\Psi = 0 \qquad\qquad W^2\Psi = \mu^2\Psi$$

and thus Ψ will describe the irreducible representation of the Poincare group with continuous spin.

Continuous spin equations

Spin-tensor representation

One can show that $\Psi(p,\xi,\bar{\xi})$ does not have a solution in the form of a series in ξ and $\bar{\xi}$. Therefore we solve one of the equations in the form

$$\Psi(p,\xi,\bar{\xi}) = \delta((\xi\sigma^{\mu}\bar{\xi})p_{\mu} - i\mu)\,\varphi(p,\xi,\bar{\xi}).$$

One can prove that if the field $\varphi(p,\xi,\bar{\xi})$ obeys equations

$$\partial^2\varphi(x,\xi,\bar{\xi}) = 0,$$
$$\left(\bar{\sigma}^{\mu\dot{a}a}\frac{\partial}{\partial\xi^{a}}\frac{\partial}{\partial\bar{\xi}^{\dot{a}}}\frac{\partial}{\partial x^{\mu}} + \mu\right)\varphi(x,\xi,\bar{\xi}) = 0$$

then the field $\Psi(p,\xi,\bar{\xi})$ will satisfy the equations defining continuous spin representation.

Continuous spin equations

Spin-tensor representation

These two equations have a solution in the form of an expansion in ξ and $\bar{\xi}$

$$\varphi(x,\xi,\bar{\xi}) = \sum_{n,k=0}^{\infty} \frac{1}{\sqrt{n!k!}} \, \varphi_{a_1\dots a_n \dot{b}_1\dots \dot{b}_k}(x) \, \xi^{a_1} \dots \xi^{a_n} \, \bar{\xi}^{\dot{b}_1} \dots \bar{\xi}^{\dot{b}_k}.$$

Since we are going to construct Lagrangian for real bosonic fields we will consider $n = k$ case.

Lagrangian construction

Following the general BRST approach in higher spin field theory we begin with realization of the equations in auxiliary Fock space

Let us introduce creation and annihilation operators

$$\langle 0|\bar{c}_{\dot{b}} = \langle 0|c^{a} = 0, \qquad \bar{a}^{\dot{b}}|0\rangle = a_a|0\rangle = 0, \qquad \langle 0|0\rangle = 1$$

with the following nonzero commutation relations

$$[\bar{a}^{\dot{\alpha}}, \bar{c}_{\dot{\beta}}] = \delta^{\dot{\alpha}}_{\dot{\beta}}, \qquad [a_\alpha, c^{\beta}] = \delta^{\beta}_{\alpha}.$$

The states in the auxiliary Fock space are defined as follows

$$|\varphi\rangle = \sum_{k=0}^{\infty} |\varphi_k\rangle \qquad |\varphi_k\rangle = \frac{1}{k!}\, \varphi_{a(k)}{}^{\dot{b}(k)}(x)\, c^{a(k)}\, \bar{c}_{\dot{b}(k)}|0\rangle.$$

We define the Hermitian conjugation in the Fock space by the rule

$$(a_a)^{+} = \bar{c}_{\dot{a}} \qquad (\bar{c}_{\dot{a}})^{+} = a_a \qquad (\bar{a}_{\dot{a}})^{+} = c_a \qquad (c_a)^{+} = \bar{a}_{\dot{a}}$$

Lagrangian construction

Then we introduce the following operators

$$l_0 = \partial^2 \qquad l_1 = a^a \partial_{a\dot{b}} \bar{a}^{\dot{b}} - \mu \qquad l_1^+ = -c^b \partial_{b\dot{a}} \bar{c}^{\dot{a}} - \mu$$

Here $\partial_{a\dot{b}} = \sigma^\mu_{a\dot{b}} \partial_\mu$. State $|\varphi\rangle$ will describe the continuous spin representation if the constraints on the vector $|\varphi\rangle$ are satisfied

$$l_0 |\varphi\rangle = 0 \qquad l_1 |\varphi\rangle = 0$$

To construct Lagrangian we must construct Hermitian BRST operator.

The set of operator underlying the BRST operator

1) must be invariant under Hermitian conjugation
2) forms an algebra
3) must not generate any additional restrictions on the physical field

Lagrangian construction

To obtain such set of operators we first add to operators l_0, l_1 the operator l_1^+. Set of operators l_0, l_1, l_1^+ is invariant under Hermitian conjugation. Moreover this set of operators form an algebra with the only nonzero commutator

$$[l_1^+, l_1] = (N + \bar{N} + 2) l_0, \qquad N = c^a a_a, \quad \bar{N} = \bar{c}_{\dot{a}} \bar{a}^{\dot{a}}$$

and does not generate any additional restrictions on $|\varphi\rangle$.

BRST operator construted on the base of operators l_0, l_1, l_1^+ is

$$Q = \eta_0 l_0 + \eta_1^+ l_1 + \eta_1 l_1^+ + \eta_1^+ \eta_1 (N + \bar{N} + 2) \mathcal{P}_0$$

Here we have extended the Fock space by introducing η_0, η_1, η_1^+ which are the fermionic ghost “coordinates” and $\mathcal{P}_0$, $\mathcal{P}_1^+$, $\mathcal{P}_1$ which are their canonically conjugated ghost “momenta” respectively.

Lagrangian construction

These operators obey the anticommutation relations

$$\{\eta_1, \mathcal{P}_1^+\} = \{\mathcal{P}_1, \eta_1^+\} = \{\eta_0, \mathcal{P}_0\} = 1$$

and act on the vacuum state as follows

$$\eta_1|0\rangle = \mathcal{P}_1|0\rangle = \mathcal{P}_0|0\rangle = 0.$$

They possess the standard ghost numbers, $gh(\eta^i) = -gh(\mathcal{P}_i) = 1$, providing the property $gh(Q) = 1$.
The equations of motion and gauge transformations are

$$Q|\Phi\rangle = 0 \qquad |\Phi'\rangle = |\Phi\rangle + Q|\Lambda\rangle$$

where

$$|\Phi\rangle = |\varphi\rangle + \eta_0\mathcal{P}_1^+|\varphi_1\rangle + \eta_1^+\mathcal{P}_1^+|\varphi_2\rangle \qquad |\Lambda\rangle = \mathcal{P}_1^+|\lambda\rangle$$

The fields $|\varphi_1\rangle$, $|\varphi_2\rangle$ and and the gauge parameter $|\lambda\rangle$ have similar decomposition like $|\varphi\rangle$.

Lagrangian construction

The Lagrangian for the continuous spin field is constructed in the framework of the BRST approach as follows

$$
\begin{aligned}
\mathcal{L} &= \int d\eta_0 \, \langle\Phi|Q|\Phi\rangle \; = \\
&= \langle\bar{\varphi}|\Big\{l_0|\varphi\rangle - l_1^+|\varphi_1\rangle\Big\} - \langle\bar{\varphi}_1|\Big\{l_1|\varphi\rangle - l_1^+|\varphi_2\rangle + (N+\bar{N}+2)|\varphi_1\rangle\Big\} \\
&\quad - \langle\bar{\varphi}_2|\Big\{l_0|\varphi_2\rangle - l_1|\varphi_1\rangle\Big\}
\end{aligned}
$$

Using the equation of motion for field $|\varphi_1\rangle$ we remove it from the Lagrangian.

The obtained Lagrangian after redefinition of the fields
$\varphi^{\mu(n)} \to A_n\phi_I^{a(n)}$, $\varphi_2^{\mu(n)} \to -A_n\phi_{II}^{a(n)}$
and gauge parameters $\lambda^{\mu(n)} \to A_{n+1}\xi^{a(n)}$ where $A_n = (2^{n+1}n!)^{-1/2}$ will be exactly the same as Metsaev's Lagrangian for continuous spin field [Phys. Lett. B **781** (2018) 568] in the case $d=4$, $m=0$.

Summary

- We have formulated equations for the irreducible representation of the Poincaré group with continuous spin in terms of two auxiliary bosonic 2-component spinor variables.
- The Lagrangian for the bosonic continuous spin field is constructed with the help of BRST method.

THE END

THANK YOU FOR YOUR ATTENTION

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