Supersymmetric Higher Spin Gauge Theories

Sergei M. Kuzenko

Department of Physics, University of Western Australia

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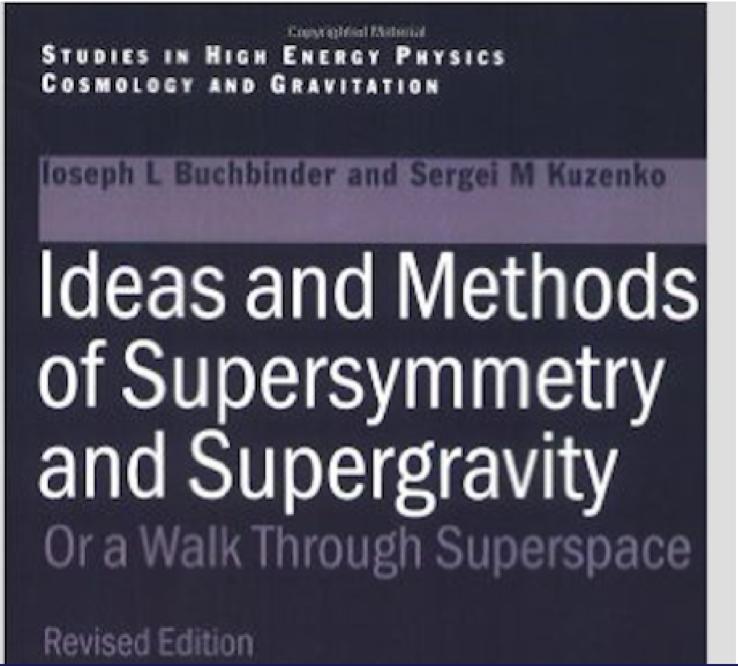
STUDIES IN HIGH ENERGY PHYSICS
COSMOLOGY AND GRAVITATION
Ioseph L Buchbinder and Sergei M Kuzenko
Ideas and Methods of Supersymmetry and Supergravity
Or a Walk Through Superspace
Revised Edition

Historial comments: Twenty five years later

$d = 4$

Off-shell formulations for massless higher-spin $\mathcal{N} = 1$ supermultiplets in Minkowski space were constructed twenty five years ago in Tomsk.

- Half-integer superspin $s + \frac{1}{2} \geq \frac{5}{2}$ (spin-$(s + \frac{1}{2})$ & spin-$(s + 1)$ fields):
  SMK, V. Postnikov & A. Sibiryakov (1993)
- Integer superspin $s \geq 2$ (spin-s & spin-$(s + \frac{1}{2})$ fields):
  SMK & A. Sibiryakov (1993)

A few months later, (off-shell) formulations for massless higher-spin $\mathcal{N} = 1$ supermultiplets in anti-de Sitter space were developed.
SMK & A. Sibiryakov (presented at SQS1993; published 1994)

No model for higher-spin supermultiplets in AdS_4 existed before 1993. Until now, however, our construction has hardly been appreciated or understood by many higher-spin practitioners, probably since it was given in terms of superfields, albeit with the key component results included.

Massless gauge superfields of higher half-integer superspins

S. M. Kuzenko, V. V. Postnikov,* and A. G. Sibiryakov
Tomsk State University, 634050 Tomsk, Russia
**L. D. Landau Institute of Theoretical Physics, Russian Academy of Sciences, 117334 Moscow, Russia*

(Submitted 31 March 1993)
Pis'ma Zh. Eksp. Teor. Fiz. **57**, No. 9, 521–525 (10 May 1993)

Two superfield formulations (dual with respect to one another) are proposed for free massless $N=1$, $D=4$ supersymmetric theories of superspin $(s+1/2)$, where $s=2,3,\ldots$. For $s=1$ the first version reduces to linearized $n=-1$ nonminimal supergravity and the second version reduces to linearized minimal supergravity.

Massless gauge superfields of higher integer superspins

S. M. Kuzenko and A. G. Sibiryakov
Tomsk State University, 634050 Tomsk, Russia

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Two (dual) superfield formulations are proposed for free massless $N=1$, $D=4$ supermultiplets $(s, s+1/2)$, where $s=1, 2, \ldots$. For $s=1$ one of these versions is identical to the nonminimal formulation for the supermultiplet $(1, 3/2)$ [E. S. Fradkin and M. Vasiliev, Nuovo Cim. Lett. **25**, 79 (1979); B. de Witt and J. W. van Holten, Nucl. Phys. B **155**, 530 (1979); S. J. Gates and W. Siegel, Nucl. Phys. B **164**, 484 (1980)].

Historical comments: Original motivations

- Late 1980s–early 1990s: Higher-spin revolution

  (E. Fradkin & M. Vasiliev)

- In 1991–1992, I was writing two chapters on $\mathcal{N} = 1$ supergravity for my book with Ioseph L Buchbinder "Walk Through Superspace" (chapters 5 & 6 in the published book).
  Project "Walk Through Superspace" was initiated by ILB.
- At the end of chapter 6, I wanted to include a section devoted to supersymmetric extensions on the (Fang-)Fronsdal models for massless higher-spin fields.
- Since every 4D $\mathcal{N} = 1$ supersymmetric theory must possess an off-shell realisation, I was specifically interested in off-shell formulations for massless higher-spin supermultiplets. Sadly, such formulations did not exist at that time.
- In the literature, there had been known two on-shell descriptions (without auxiliary fields) of the massless higher-spin supermultiplets:

  T. Curtright (1979)
  M. Vasiliev (1980)

Historical comments: On-shell approaches

The Curtright formulation

(Example of "jump on it" research, terminology due to Ian McArthur)
By putting together two (Fang-)Fronsdal models for massless spin-$\hat{s}$ and spin-$(\hat{s}+\frac{1}{2})$ fields, with $2\hat{s} \geq 4$ an integer, he showed that the gauge-invariant action possessed on-shell SUSY (i.e., the algebra of supersymmetry transformations closes on the mass shell only).

- The Fronsdal model for a massless field of integer spin $s \geq 2$ (1978) is described in terms of two real fields:
  $h_{\alpha(s)\dot{\alpha}(s)} := h_{\alpha_1\dots\alpha_s\dot{\alpha}_1\dots\dot{\alpha}_s} = h_{(\alpha_1\dots\alpha_s)(\dot{\alpha}_1\dots\dot{\alpha}_s)}$ (conformal) gauge field
  $\varphi_{\alpha(s-2)\dot{\alpha}(s-2)}$ compensator
- The Fang-Fronsdal model for a massless spin-$(s+\frac{1}{2})$ field (1978) is described in terms of three complex fields:
  $\psi_{\alpha(s+1)\dot{\alpha}(s)}$ (conformal) gauge field
  $\chi_{\alpha(s-1)\dot{\alpha}(s)}$ compensator
  $\rho_{\alpha(s-1)\dot{\alpha}(s-2)}$ compensator
  & their conjugates.

Historical comments

- Gauge freedom in the Fronsdal model

$$\delta h_{\alpha_1\dots\alpha_s\dot\alpha_1\dots\dot\alpha_s} = \partial_{(\alpha_1(\dot\alpha_1}\zeta_{\alpha_2\dots\alpha_s)\dot\alpha_2\dots\dot\alpha_s)} ,$$
$$\delta\varphi_{\alpha_1\dots\alpha_{s-2}\dot\alpha_1\dots\dot\alpha_{s-2}} = \partial^{\beta\dot\beta}\zeta_{\beta\alpha_1\dots\alpha_{s-2}\dot\beta\dot\alpha_1\dots\dot\alpha_{s-2}} .$$

- Gauge freedom in the Fang-Fronsdal model

$$\delta\psi_{\alpha_1\dots\alpha_{s+1}\dot\alpha_1\dots\dot\alpha_s} = \partial_{(\alpha_1(\dot\alpha_1}\xi_{\alpha_2\dots\alpha_{s+1})\dot\alpha_2\dots\dot\alpha_s)} ,$$
$$\delta\chi_{\alpha_1\dots\alpha_{s-1}\dot\alpha_1\dots\dot\alpha_s} = \partial^{\beta}{}_{(\dot\alpha_1}\xi_{\beta\alpha_2\dots\alpha_{s-1}\dot\alpha_2\dots\dot\alpha_s)} ,$$
$$\delta\rho_{\alpha_1\dots\alpha_{s-1}\dot\alpha_1\dots\dot\alpha_{s-2}} = \partial^{\beta\dot\beta}\xi_{\beta\alpha_1\dots\alpha_{s-1}\dot\beta\dot\alpha_1\dots\dot\alpha_{s-2}} .$$

Side remark: Conformal higher-spin gauge theory

E. Fradkin & A. Tseytlin (1985)

Throw away the compensators and construct an action in terms of conformal primary and gauge-invariant field strengths, such as

$$\mathcal{C}_{\alpha_1\dots\alpha_{2s}} = \partial_{(\alpha_1}{}^{\dot\beta_1}\dots\partial_{\alpha_s}{}^{\dot\beta_s} h_{\alpha_{s+1}\dots\alpha_{2s})\dot\beta_1\dots\dot\beta_s}$$

Historical comments: Curtright's formulation

- On general grounds, given any actions for free massless spin-$\hat{s}$ and spin-$(\hat{s} + \frac{1}{2})$ fields, their sum should possess some on-shell supersymmetry. In this sense, there is no problem, whatsoever, of constructing on-shell massless higher spin supermultiplets. One only needs to work out supersymmetry transformations.
- Curtright's construction offered the first Lagrangian model for massless superspin-$\hat{s}$ multiplet – in this sense, truly important work. Otherwise there is nothing special/unique in his approach, in my opinion. In particular, Vasiliev's construction is more general.

Historical comments: Vasiliev's formulation

Vasiliev pioneered the so-called frame-like formulation for massless higher-spin fields as an extensions of that for gravity.

- Description of the gravitational field ($s = 2$) in terms of one-forms:
  $e^a = \mathrm{d}x^m e_m{}^a$ vielbein
  $\omega_{ab} = -\omega_{ba} = \mathrm{d}x^m \omega_{m,ab}$ Lorentz connection
- Integer spin $s > 2$
  $e^{a(s-1)} = \mathrm{d}x^m e_m{}^{a_1 \dots a_{s-1}}$ generalised vielbein
  symmetric and traceless in the a-indices
  $\omega_{b,a(s-1)} = \mathrm{d}x^m \omega_{m,b,a(s-1)}$ generalised connection
  $\omega_{b,a(s-1)}$ is symmetric and traceless in the a indices and obey the additional properties $\omega_{a,a(s-1)} = 0$, $\omega^a{}_{,a(s-1)} = 0.$
  The connection should be eliminated algebraically in terms of $e_m{}^{a_1 \dots a_{s-1}}$ and its derivatives on the mass shell.
  Intelligent spinor description: $e_m{}^{a_1 \dots a_{s-1}} \rightarrow e_{m,\alpha(s-1)\dot\alpha(s-1)}$
- Half-integer spin $s + \frac{1}{2}$, $s > 2$
  $\psi_{m,\alpha(s)\dot\alpha(s-1)}$, $\bar\psi_{m,\alpha(s-1)\dot\alpha(s)}$ in conjunction with an algebraic gauge symmetry that is destined to kill $\psi_\alpha{}^{\dot\alpha}{}_{,\alpha(s)\dot\alpha(s-1)}$.

Historical comments: Lessons from supergravity

- In all off-shell formulations for supergravity-matter systems (Wess-Zumino approach, superconformal tensor calculus, U($\mathcal{N}$) superspace, conformal superspace), one never works with an independent Lorentz connection. It is fixed, once and forever, by imposing covariant constraints.
- In the higher-spin case, it is therefore desirable to look for a formalism that does not make use of an independent generalised connection $\omega_{b,a(s-1)} = \mathrm{d}x^m \omega_{m,b,a(s-1)}$.
- In $\mathcal{N} = 1$ supergravity (massless spin-$(\frac{3}{2}, 2)$ supermultiplet), the inverse vielbein is embedded in Ferrara-Zumino-Ogievetsky-Sokatchev-Siegel superfield

$$H^m(\theta, \bar{\theta}) = \dots + \theta\sigma^a\bar{\theta} e_a{}^m + \bar{\theta}^2\theta^\alpha\psi_\alpha{}^m + \theta^2\bar{\theta}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}m} + \theta^2\bar{\theta}^2 A^m$$

- Massless $\mathcal{N} = 1$ gravitino supermultiplet, $(1, \frac{3}{2})$, is described by Ogievetsky-Sokatchev spinor superfield

$$\Psi_\alpha(\theta, \bar{\theta}) = \dots + \theta\sigma^b\bar{\theta}\psi_{b,\alpha} + \dots$$

Historical comments

Early attempts to construct off-shell superfield realisations for massless higher-spin supermultiplets by

M. Bellon & S. Ouvry (1987)

was unsuccessful.

- The superfield formulation proposed does not correspond to the specific features of genuinely massless higher-spin supermultiplets.
- Supermultiplets arising within this scheme are too reducible, and additional constraints should be imposed in order to eliminate extra degrees of freedom.
- Such constraints were found by Bellon & Ouvry only for the gravitino multiplet; however, correct off-shell descriptions of this multiplet had been given much earlier.
- This formulation cannot be lifted to AdS supersymmetry.

Conformal gauge superfields: Half-integer superspin

Let s be a positive integer. In the superspin-$(s+\frac{1}{2})$ case, the conformal prepotential $H_{\alpha(s)\dot\alpha(s)} \equiv H_{\alpha_1\dots\alpha_s\dot\alpha_1\dots\dot\alpha_s}$ is a real superfield, which is symmetric in its undotted indices and, independently, in its dotted indices. The gauge transformation law of $H_{\alpha(s)\dot\alpha(s)}$ is

$$\delta H_{\alpha_1\dots\alpha_s\dot\alpha_1\dots\dot\alpha_s} = \bar D_{(\dot\alpha_1}\Lambda_{\alpha_1\dots\alpha_s\dot\alpha_2\dots\dot\alpha_s)} - D_{(\alpha_1}\bar\Lambda_{\alpha_2\dots\alpha_s)\dot\alpha_1\dots\dot\alpha_s} ,$$

with unconstrained gauge parameter $\Lambda_{\alpha(s)\dot\alpha(s-1)}$.

P. Howe, K. Stelle & P Townsend (1981)
SMK, V. Postnikov & A. Sibiryakov (1993)

The $s=1$ case corresponds to linearised conformal supergravity

S. Ferrara and B. Zumino (1978)

Wess-Zumino gauge:

$$H_{\alpha(s)\dot\alpha(s)}(\theta,\bar\theta) = \theta^\beta\bar\theta^{\dot\beta}e_{\beta\dot\beta,\alpha(s)\dot\alpha(s)} + \bar\theta^2\theta^\beta\psi_{\beta,\alpha(s)\dot\alpha(s)} - \theta^2\bar\theta^{\dot\beta}\bar\psi_{\alpha(s)\dot\beta,\dot\alpha(s)} + \theta^2\bar\theta^2 A_{\alpha_1\dots\alpha_s\dot\alpha_1\dots\dot\alpha_s} .$$

Here $e_{\beta\dot\beta,\alpha(s)\dot\alpha(s)}$ may be identified with Vasiliev's generalised vielbein.

Conformal gauge superfields: Integer superspin

In the superspin-s case, s integer, superconformal multiplet is described by unconstrained prepotential $\Psi_{\alpha(s)\dot\alpha(s-1)} \equiv \Psi_{\alpha_1\dots\alpha_s\dot\alpha_1\dots\dot\alpha_{s-1}}$ and its conjugate $\bar\Psi_{\alpha(s-1)\dot\alpha(s)}$.
For $s > 1$ the gauge freedom is

$$\delta\Psi_{\alpha_1\dots\alpha_s\dot\alpha_1\dots\dot\alpha_{s-1}} = D_{(\alpha_1}\bar\Lambda_{\alpha_2\dots\alpha_s)\dot\alpha_1\dots\dot\alpha_{s-1}} + \bar D_{(\dot\alpha_1}\zeta_{\alpha_1\dots\alpha_s\dot\alpha_2\dots\dot\alpha_{s-1})} ,$$

with unconstrained gauge parameters $\bar\Lambda_{\alpha(s-1)\dot\alpha(s-1)}$ and $\zeta_{\alpha(s)\dot\alpha(s-2)}$.
$\bar\Lambda_{\alpha(s-1)\dot\alpha(s-1)} = D^\beta L_{(\alpha_1\dots\alpha_{s-1}\beta)\dot\alpha(s-1)}$

SMK & A. Sibiryakov (1993)
SMK, R. Manvelyan & S. Theisen (2017)

Wess-Zumino gauge

$$\Psi_{\alpha(s)\dot\alpha(s-1)}(\theta,\bar\theta) = \theta^\beta\bar\theta^{\dot\beta}\psi_{\beta\dot\beta,\alpha(s)\dot\alpha(s-1)} + \bar\theta^2\theta^\beta B_{\beta,\alpha(s)\dot\alpha(s-1)} - \theta^2\bar\theta^{\dot\beta}C_{\alpha(s)\dot\beta,\dot\alpha(s-1)} + \theta^2\bar\theta^2\Upsilon_{\alpha(s)\dot\alpha(s-1)} .$$

Here $\psi_{\beta\dot\beta,\alpha(s)\dot\alpha(s-1)}$ may be identified with Vasiliev's spin-$(s+\frac{1}{2})$ gauge field. The relevant algebraic gauge symmetry comes from $\zeta_{\alpha(s)\dot\alpha(s-2)}$.

Lessons from off-shell supergravity (superspin 3/2)

- Every off-shell formulation for $\mathcal{N}=1$ supergravity can be realised as conformal supergravity coupled to a compensating supermultiplet. Different compensators lead to different supergravity theories.
- At the linearised level, conformal supergravity is described by real a prepotential $H_{\alpha\dot{\alpha}}$ with the gauge freedom

$$\delta H_{\alpha\dot{\alpha}} = \bar{D}_{\dot{\alpha}}\Lambda_{\alpha} - D_{\alpha}\bar{\Lambda}_{\dot{\alpha}} \ .$$

- Old minimal supergravity: chiral compensator σ, $\bar{D}_{\dot{\alpha}}\sigma = 0$.

$$\delta\sigma = -\frac{1}{12}\bar{D}^2 D^{\alpha}\Lambda_{\alpha} \ .$$

- New minimal supergravity: real linear compensator $L=\bar{L}$, $\bar{D}^2 L = 0$.

$$\delta L = \frac{1}{4}(D^{\alpha}\bar{D}^2\Lambda_{\alpha} + \bar{D}_{\dot{\alpha}}D^2\bar{\Lambda}^{\dot{\alpha}}) \ .$$

- Non-minimal supergravity: complex linear compensator Γ, $\bar{D}^2\Gamma = 0$.

$$\delta\Gamma = -\frac{1}{4}\frac{n+1}{3n+1}\bar{D}^2 D^{\alpha}\Lambda_{\alpha} - \frac{1}{4}\bar{D}_{\dot{\alpha}}D^2\bar{\Lambda}^{\dot{\alpha}} \ .$$

Massless gauge supermultiplets: Half-integer superspin

An off-shell description of a massless multiplet of half-integer superspin $s + \frac{1}{2} \geq \frac{5}{2}$ should involve the conformal prepotential $H_{\alpha(s)\dot{\alpha}(s)}$ coupled to a compensating supermultiplet. This was clear to me at the end of 1991. The main problem was to identify compensators. Final solution:

SMK, V. Postnikov & A. Sibiryakov (1993)

- Transverse formulation

$$\mathcal{V}^{\perp}_{(s+\frac{1}{2})} = \left\{ H_{\alpha(s)\dot{\alpha}(s)} \ , \ \Gamma_{\alpha(s-1)\dot{\alpha}(s-1)} \ , \ \bar{\Gamma}_{\alpha(s-1)\dot{\alpha}(s-1)} \right\} ,$$

where complex superfield $\Gamma_{\alpha(s-1)\dot{\alpha}(s-1)}$ is transverse linear

$$\bar{D}^{\dot{\beta}} \Gamma_{\alpha(s-1)\dot{\beta}\dot{\alpha}(s-2)} = 0 \quad \Longrightarrow \quad \bar{D}^2 \Gamma_{\alpha(s-1)\dot{\alpha}(s-1)} = 0 \ .$$

- Longitudinal formulation

$$\mathcal{V}^{\parallel}_{(s+\frac{1}{2})} = \left\{ H_{\alpha(s)\dot{\alpha}(s)} \ , \ G_{\alpha(s-1)\dot{\alpha}(s-1)} \ , \ \bar{G}_{\alpha(s-1)\dot{\alpha}(s-1)} \right\}$$

where complex superfield $G_{\alpha(s-1)\dot{\alpha}(s-1)}$ is longitudinal linear

$$\bar{D}_{(\dot{\alpha}_1} G_{\alpha(s-1)\dot{\alpha}_2 \dots \dot{\alpha}_s)} = 0 \quad \Longrightarrow \quad \bar{D}^2 G_{\alpha(s-1)\dot{\alpha}(s-1)} = 0 \ .$$

Transverse linear and longitudinal linear supermultiplets

Complex tensor superfields $\Gamma_{\alpha(m)\dot\alpha(n)}$ and $G_{\alpha(m)\dot\alpha(n)}$ are said to be transverse linear and longitudinal linear, respectively, if the constraints

$$\begin{aligned}
\bar D^{\dot\beta}\Gamma_{\alpha(m)\dot\beta\dot\alpha(n-1)} &= 0\,, \qquad n\neq 0 \quad &\Longrightarrow \quad \bar D^2\Gamma_{\alpha(m)\dot\alpha(n)} = 0\\
\bar D_{(\dot\alpha_1}G_{\alpha(m)\dot\alpha_2\dots\dot\alpha_{n+1})} &= 0 &\Longrightarrow \quad \bar D^2 G_{\alpha(m)\dot\alpha(n)} = 0
\end{aligned}$$

are satisfied. These constraints are solved in terms of unconstrained prepotentials $\Phi_{\alpha(m)\dot\alpha(n+1)}$ and $\Psi_{\alpha(m)\dot\alpha(n-1)}$ as follows:

$$\begin{aligned}
\Gamma_{\alpha(m)\dot\alpha(n)} &= \bar D^{\dot\beta}\Phi_{\alpha(m)\,(\dot\beta\dot\alpha_1\cdots\dot\alpha_n)}\,,\\
G_{\alpha(m)\dot\alpha(n)} &= \bar D_{(\dot\alpha_1}\Psi_{\alpha(m)\,\dot\alpha_2\cdots\dot\alpha_n)}\,,
\end{aligned}$$

These prepotentials are defined modulo gauge transformations

$$\begin{aligned}
\delta_\xi\Phi_{\alpha(m)\,\dot\alpha(n+1)} &= \bar D^{\dot\beta}\xi_{\alpha(m)\,(\dot\beta\dot\alpha_1\cdots\dot\alpha_{n+1})}\,,\\
\delta_\zeta\Psi_{\alpha(m)\,\dot\alpha(n-1)} &= \bar D_{(\dot\alpha_1}\zeta_{\alpha(m)\,\dot\alpha_2\cdots\dot\alpha_{n-1})}\,,
\end{aligned}$$

with the parameters $\xi_{\alpha(m)\,\dot\alpha(n+2)}$ and $\zeta_{\alpha(m)\,\dot\alpha(n-2)}$ being unconstrained.

Transverse linear and longitudinal linear supermultiplets

There emerge the transverse and longitudinal gauge hierarchies:

$$\Gamma_{\dot\alpha(n)} \longrightarrow \Gamma_{\dot\alpha(n+1)} \longrightarrow \Gamma_{\dot\alpha(n+2)} \cdots$$
$$G_{\dot\alpha(n)} \longrightarrow G_{\dot\alpha(n-1)} \longrightarrow G_{\dot\alpha(n-2)} \cdots \longrightarrow G~.$$

In accordance with the terminology of gauge theories with linearly dependent generators (Batalin-Vilkovisky quantisation), any Lagrangian theory described by a transverse (longitudinal) linear superfield $\Gamma_{\dot\alpha(n)}$ $(G_{\dot\alpha(n)})$ can be considered as the theory of an unconstrained prepotential $\xi_{\dot\alpha(n+1)}$ $(\zeta_{\dot\alpha(n-1)})$ with an additional gauge invariance of infinite (finite) stage of reducibility.

Transverse linear and longitudinal linear superfields were pioneered by E. Ivanov & A. Sorin (1980) in their analysis of the representations of 4D $\mathcal{N}=1$ AdS supersymmetry.

Massless gauge supermultiplets: Half-integer superspin

Gauge freedom

$$\begin{aligned}
\delta_\Lambda H_{\alpha_1\dots\alpha_s\dot\alpha_1\dots\dot\alpha_s} &= \bar D_{(\dot\alpha_1}\Lambda_{\alpha_1\dots\alpha_s\dot\alpha_2\dots\dot\alpha_s)} - D_{(\alpha_1}\bar\Lambda_{\alpha_2\dots\alpha_s)\dot\alpha_1\dots\dot\alpha_s}~, \\
\delta_\Lambda \Gamma_{\alpha_1\dots\alpha_{s-1}\dot\alpha_1\dots\dot\alpha_{s-1}} &= -\frac{1}{4}\bar D^{\dot\beta} D^2 \bar\Lambda_{\alpha_1\dots\alpha_{s-1}\dot\beta\dot\alpha_1\dots\dot\alpha_{s-1}}~, \\
\delta_\Lambda G_{\alpha_1\dots\alpha_{s-1}\dot\alpha_1\dots\dot\alpha_{s-1}} &= -\frac{1}{2}\bar D_{(\dot\alpha_1}\bar D^{|\dot\beta|} D^\beta \Lambda_{\beta\alpha_1\dots\alpha_{s-1}\dot\alpha_2\dots\dot\alpha_{s-1})\dot\beta} \\
&\quad + \mathrm{i}(s-1)\bar D_{(\dot\alpha_1}\partial^{\beta|\dot\beta|}\Lambda_{\beta\alpha_1\dots\alpha_{s-1}\dot\alpha_2\dots\dot\alpha_{s-1})\dot\beta}~.
\end{aligned}$$

Here the gauge parameter $\Lambda_{\alpha(s)\dot\alpha(s-1)}$ is unconstrained.

Gauge transformation laws of the prepotentials:

$$\begin{aligned}
\delta_\Lambda \Phi_{\alpha_1\dots\alpha_{s-1}\dot\alpha_1\dots\dot\alpha_s} &= -\frac{1}{4} D^2 \bar\Lambda_{\alpha_1\dots\alpha_{s-1}\dot\alpha_1\dots\dot\alpha_s}~, \\
\delta_\Lambda \Psi_{\alpha_1\dots\alpha_{s-1}\dot\alpha_1\dots\dot\alpha_{s-2}} &= -\frac{1}{2}\Big(\bar D^{\dot\beta} D^\beta - 2\mathrm{i}(s-1)\partial^{\beta\dot\beta}\Big)\Lambda_{\beta\alpha_1\dots\alpha_{s-1}\dot\beta\dot\alpha_1\dots\dot\alpha_{s-2}}~.
\end{aligned}$$

Transverse linear – longitudinal linear duality

Consider a supersymmetric theory described by a transverse linear superfield $\Gamma_{\alpha(m)\dot\alpha(n)}$ and its conjugate $\bar\Gamma_{\alpha(n)\dot\alpha(m)}$, for $n > 0$, with an action functional $S[\Gamma, \bar\Gamma]$. Such a theory possesses a dual formulation, $S_{\rm D}[G, \bar G]$, described in terms of a longitudinal linear superfield $G_{\alpha(n)}$ and its conjugate $\bar G_{\alpha(n)}$ It is obtained by introducing a first-oder action

$$S[V, \bar V, G, \bar G] = S[V, \bar V] + \int {\rm d}^4x{\rm d}^2\theta{\rm d}^2\bar\theta \left(V^{\alpha(m)\dot\alpha(n)} G_{\alpha(m)\dot\alpha(n)} + {\rm c.c} \right) ,$$

with $V_{\alpha(m)\dot\alpha(n)}$ being unconstrained complex. Here $S[V, \bar V]$ is obtained from $S[\Gamma, \bar\Gamma]$ by the replacement $\Gamma_{\alpha(m)\dot\alpha(n)} \to V_{\alpha(m)\dot\alpha(n)}$. Varying $S[V, \bar V, G, \bar G]$ with respect to $G_{\alpha(m)\dot\alpha(n)}$ gives $V_{\alpha(m)\dot\alpha(n)} = \Gamma_{\alpha(m)\dot\alpha(n)}$, and then $S[V, \bar V, G, \bar G]$ reduces to $S[\Gamma, \bar\Gamma]$. On the other hand, we can integrate out the auxiliary superfield $V_{\alpha(m)\dot\alpha(n)}$ by making use of its equation of motion

$$\frac{\delta}{\delta V^{\alpha(m)\dot\alpha(n)}} S[V, \bar V] + G_{\alpha(m)\dot\alpha(n)} = 0 .$$

This leads to the dual action $S_{\rm D}[G, \bar G]$.

Massless gauge supermultiplets: Integer superspin

Massless multiplet of integer superspin $s \geq 2$

SMK & A. Sibiryakov (1993)

- Transverse formulation

$$\mathcal{V}^{\perp}_{(s)} = \Big\{ H_{\alpha(s-1)\dot\alpha(s-1)} \,,\; \Gamma_{\alpha(s)\dot\alpha(s)} \,,\; \bar\Gamma_{\alpha(s)\dot\alpha(s)} \Big\} \,.$$

- Longitudinal formulation

$$\mathcal{V}^{\parallel}_{(s)} = \Big\{ H_{\alpha(s-1)\dot\alpha(s-1)} \,,\; G_{\alpha(s)\dot\alpha(s)} \,,\; \bar G_{\alpha(s)\dot\alpha(s)} \Big\} \,.$$

Gauge freedom:

$$\begin{aligned}
\delta H_{\alpha(s-1)\dot\alpha(s-1)} &= D^{\beta} L_{\beta\alpha(s-1)\dot\alpha(s-1)} - \bar D^{\dot\beta} \bar L_{\alpha(s-1)\dot\beta\dot\alpha(s-1)} \,, \\
\delta \Gamma_{\alpha(s)\dot\alpha(s)} &= \frac{s+1}{2(s+2)} \bar D^{\dot\gamma} \bar D_{(\dot\gamma} D_{(\alpha_1} \bar L_{\alpha_2\dots\alpha_s)\dot\alpha(s))} \\
&\quad + \mathrm{i}(s+1) \bar D^{\dot\gamma} D_{(\alpha_1(\dot\gamma} \bar L_{\alpha_2\dots\alpha_s)\dot\alpha(s))} \,, \\
\delta G_{\alpha(s)\dot\alpha(s)} &= \frac{1}{2} \bar D_{(\dot\alpha_1} D_{(\alpha_1} \mathcal{D}^{|\gamma|} L_{\alpha_2\dots\alpha_s)\gamma\dot\alpha_2\dots\dot\alpha_s)} \,,
\end{aligned}$$

where the gauge parameter $L_{\alpha(s)\dot\alpha(s-1)}$ is unconstrained complex.

Massless gauge supermultiplets: Integer superspin

In terms of the complex unconstrained prepotential $\Psi_{\alpha(s)\dot\alpha(s-1)}$,

$$G_{\alpha(s)\dot\alpha(s)} = \bar D_{(\dot\alpha_1}\Psi_{\alpha_1\dots\alpha_s\dot\alpha_2\dots\dot\alpha_s)} ~,$$

the gauge freedom is given by

$$\delta\Psi_{\alpha(s)\dot\alpha(s-1)} = \frac{1}{2}D_{(\alpha_1}D^{|\gamma|}L_{\alpha_2\dots\alpha_s)\gamma\dot\alpha(s-1)} + \bar D_{(\dot\alpha_1}\zeta_{\alpha_1\dots\alpha_s\dot\alpha_2\dots\dot\alpha_{s-1})} ~.$$

This differs from the transformation law of the superconformal gauge superfield

$$\delta_{\mathfrak{V},\zeta}\Psi_{\alpha(s)\dot\alpha(s-1)} = \frac{1}{2}D_{(\alpha_1}\mathfrak{V}_{\alpha_2\dots\alpha_s)\dot\alpha_1\dots\dot\alpha_{s-1}} + \bar D_{(\dot\alpha_1}\zeta_{\alpha_1\dots\alpha_s\dot\alpha_2\dots\dot\alpha_{s-1})} ~,$$

with unconstrained gauge parameters $\mathfrak{V}_{\alpha(s-1)\dot\alpha(s-1)}$ and $\zeta_{\alpha(s)\dot\alpha(s-2)}$.

Re-formulation of the longitudinal superspin-s model was given by
Poincaré SUSY — J. Hutomo & SMK (2018)
AdS SUSY — E. Buchbinder, J. Hutomo & SMK (2018)

Massless gauge supermultiplets: Integer superspin

New dynamical variables:
(i) unconstrained prepotential $\Psi_{\alpha(s)\dot\alpha(s-1)}$ and its complex conjugate;
(ii) real superfield $H_{\alpha(s-1)\dot\alpha(s-1)} = \bar H_{\alpha(s-1)\dot\alpha(s-1)}$; and
(iii) unconstrained prepotential $Z_{\alpha(s-1)\dot\alpha(s-1)}$ and its conjugate.
Gauge freedom:

$$\begin{aligned}
\delta_{\mathfrak{V},\zeta}\Psi_{\alpha_1\dots\alpha_s\dot\alpha_1\dots\dot\alpha_{s-1}} &= \frac{1}{2}\mathcal{D}_{(\alpha_1}\mathfrak{V}_{\alpha_2\dots\alpha_s)\dot\alpha_1\dots\dot\alpha_{s-1}} + \bar{\mathcal{D}}_{(\dot\alpha_1}\zeta_{\alpha_1\dots\alpha_s\dot\alpha_2\dots\dot\alpha_{s-1})} ~, \\
\delta_{\mathfrak{V}} H_{\alpha(s-1)\dot\alpha(s-1)} &= \mathfrak{V}_{\alpha(s-1)\dot\alpha(s-1)} + \bar{\mathfrak{V}}_{\alpha(s-1)\dot\alpha(s-1)} ~, \\
\delta_{\mathfrak{V},\xi} Z_{\alpha(s-1)\dot\alpha(s-1)} &= \bar{\mathfrak{V}}_{\alpha(s-1)\dot\alpha(s-1)} + \bar{\mathcal{D}}^{\dot\beta}\xi_{\alpha(s-1)(\dot\beta\dot\alpha_1\dots\dot\alpha_{s-1})} ~.
\end{aligned}$$

ξ-symmetry implies that $Z_{\alpha(s-1)\dot\alpha(s-1)}$ enters the action via the transverse linear field strength

$$\Sigma_{\alpha(s-1)\dot\alpha(s-2)} = \bar{\mathcal{D}}^{\dot\beta} Z_{\alpha(s-1)(\dot\beta\dot\alpha_1\dots\dot\alpha_{s-2})} ~.$$

Higher-spin lesson for supergravity

- In 1994, Sibiryakov and I constructed two infinite series of dually equivalent models, $S^{||}_{(s+\frac{1}{2})}$ and $S^{\perp}_{(s+\frac{1}{2})}$, describing off-shell massless gauge supermultiplets in AdS_4 of half-integer superspin $(s+\frac{1}{2})$, where $s = 1, 2 \dots$.
- For $s = 1$, the actions $S^{||}_{(3/2)}$ and $S^{\perp}_{(3/2)}$ should correspond to linearised supergravity. Indeed, the functional $S^{||}_{(3/2)}$ was known to be the linearised action for minimal $\mathcal{N} = 1$ AdS supergravity. On general grounds, $S^{\perp}_{(3/2)}$ should correspond to linearisation of a non-minimal $(n = -1)$ supergravity around an AdS background.
- However, for many years it was believed that there is no non-minimal formulation for $\mathcal{N} = 1$ AdS supergravity: S. J. Gates Jr., M. T. Grisaru, M. Roček and W. Siegel, *Superspace, or One Thousand and One Lessons in Supersymmetry*, Benjamin/Cummings (Reading, MA), 1983.
- This story puzzled me for almost twenty years.

Lesson for supergravity

- One day in 2011, I explained the puzzle to Daniel Butter, who was my postdoc at the time. The same day, we also discussed the complex linear Goldstino superfield proposed shortly before in my work with Simon Tyler (arXiv:1102.3042). The coupling of this superfield to supergravity was described by constraint $-\frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)\Gamma = f = \mathrm{const}$, plus nonlinear constraints.
- A few days later, Daniel came back with a beautiful solution to the puzzle. Consistent non-minimal $\mathcal{N} = 1$ AdS supergravity was formulated in:
  D. Butter and S. M. Kuzenko, "A dual formulation of supergravity-matter theories," Nucl. Phys. B **854**, 1 (2012) [arXiv:1106.3038 [hep-th]].
- This was achieved by demanding the conformal compensator Γ to obey the *super-Weyl invariant* constraint

$$-\frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)\Gamma = \mu = \mathrm{const}~, \qquad \mu \neq 0$$

describing a deformed complex linear superfield.

Higher-spin supercurrent multiplets

The off-shell formulations for massless higher spin supermultiplets in AdS allows one to define consistent higher spin supercurrent multiplets in AdS (i.e. higher spin extensions of the supercurrent) that contain ordinary bosonic and fermionic conserved currents in AdS.

Half-integer superspin

Let us couple the prepotentials $H_{\alpha(s)\dot\alpha(s)}$, $\Psi_{\alpha(s-1)\dot\alpha(s-2)}$ and $\bar\Psi_{\alpha(s-2)\dot\alpha(s-1)}$, to external sources

$$S^{(s+\frac12)}_{\rm source} = \int {\rm d}^4x{\rm d}^2\theta{\rm d}^2\bar\theta\, E\,\Big\{ H^{\alpha(s)\dot\alpha(s)} J_{\alpha(s)\dot\alpha(s)} + \Psi^{\alpha(s-1)\dot\alpha(s-2)} T_{\alpha(s-1)\dot\alpha(s-2)} + \bar\Psi_{\alpha(s-2)\dot\alpha(s-1)} \bar T^{\alpha(s-2)\dot\alpha(s-1)} \Big\} ~.$$

Requiring $S^{(s+\frac12)}_{\rm source}$ to be invariant under the gauge transformations gives

$$\bar{\mathcal D}^{\dot\beta} J_{\alpha_1\dots\alpha_s\dot\beta\dot\alpha_1\dots\dot\alpha_{s-1}} + \frac12 \Big(\mathcal{D}_{(\alpha_1}\bar{\mathcal D}_{(\dot\alpha_1} - 2{\rm i}(s-1)\mathcal{D}_{(\alpha_1(\dot\alpha_1}\Big) T_{\alpha_2\dots\alpha_s)\dot\alpha_2\dots\dot\alpha_{s-1})} = 0~,$$

$$\bar{\mathcal D}^{\dot\beta} T_{\alpha(s-1)\dot\beta\dot\alpha_1\dots\dot\alpha_{s-3}} = 0~.$$

$J_{\alpha(s)\dot\alpha(s)}$ higher-spin supercurrent, $T_{\alpha(s-1)\dot\alpha(s-2)}$ trace multiplet.

Works on off-shell higher-spin supermultiplets

- Off-shell formulations for massless higher-spin $\mathcal{N} = 1$ supermultiplets in $\mathbb{M}^4$

  SMK, V. Postnikov & A. Sibiryakov (1993)

- Off-shell formulations for massless higher-spin $\mathcal{N} = 1$ supermultiplets in AdS_4

  SMK & A. Sibiryakov (1993)

- Covariant quantisation of massless higher-spin supersymmetric theories in AdS_4

  I. Buchbinder, SMK & A. Sibiryakov (1995)

- Massless higher-spin $\mathcal{N} = 2$ supermultiplets in $\mathbb{M}^4$ and AdS_4

  J. Gates, SMK & A. Sibiryakov (1997)

- Generating formulation for the massless higher-spin supermultiplets in AdS_4

  J. Gates, SMK & A. Sibiryakov (1997)

Works on off-shell higher-spin supermultiplets

- Off-shell formulations for massless and topologically massive $\mathcal{N} = 2$ higher-spin supermultiplets in $\mathbb{M}^3$

  SMK & D. Ogburn (2016)

- Off-shell formulations for massless and topologically massive $\mathcal{N} = 1$ higher-spin supermultiplets in $\mathbb{M}^3$

  SMK & M. Tsulaia (2017)

- Higher-spin supercurrent multiplets in $\mathbb{M}^4$ and AdS_4

  E. Buchbinder, J. Hutomo & SMK (2018)

- Higher-spin supercurrent multiplets in $\mathbb{M}^4$ from superfield Noether procedure

  I. Buchbinder, J. Gates & K. Koutrolikos (2017–18)

- Off-shell formulations for massless and topologically massive higher-spin supermultiplets in (1,1) AdS_3 superspace

  J. Hutomo & SMK (2018)

- 4D $\mathcal{N} = 1$ superconformal higher spin multiplets

  SMK, R. Manvelyan & S. Theisen (2017)