Quantum antibrackets

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Based on

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Covariant quantization of general gauge theories - Batalin-Vilkovisky (BV) formalism (or field-antifield formalism) (I.A. Batalin, G.A. Vilkovisky, Phys.Lett. B (1981); Phys.Rev.D (1983))

Quantum master equation

$$\frac{1}{2}(W,W) = i\hbar\Delta W, \quad \Delta^2 = 0, \quad \Delta\exp\{\frac{i}{\hbar}W\} = 0.$$

Ward identity for the effective action

$$(\Gamma,\Gamma) = 0.$$

Antibracket

$$(F,G) = F\Big(\frac{\overleftarrow{\delta}}{\delta\Phi^A}\frac{\delta}{\delta\Phi^*_A} - \frac{\overleftarrow{\delta}}{\delta\Phi^*_A}\frac{\delta}{\delta\Phi^A}\Big)G, \quad \varepsilon(\Phi^A) = \varepsilon(\Phi^*_A) + 1 = \varepsilon_A.$$

Delta-operator

$$\Delta = (-1)^{\varepsilon_A}\frac{\delta}{\delta\Phi^A}\frac{\delta}{\delta\Phi^*_A}.$$

Basic properties of antibrackets:
Antisymmetry

$$(F,G) = -(G,F)(-1)^{(\varepsilon(F)+1)(\varepsilon(G)+1)}$$

Leibniz rule

$$(FG,H) = F(G,H) + (F,H)G(-1)^{\varepsilon(G)(\varepsilon(H)+1)}$$

Jacobi identity

$$(F,(G,H))(-1)^{(\varepsilon(F)+1)(\varepsilon(H)+1)} + \mathrm{cycle}(F,G,H) = 0$$

The antibracket can be considered as an odd counterpart to the Poisson bracket.

$$\text{Poisson bracket } \{\ ,\ \} \quad \rightarrow \quad \text{Commutator } [\ ,\ ]$$

$$\text{Antibracket } (\ ,\ ) \quad \rightarrow \quad ?$$

The answer is "*Quantum antibrackets*", $(\ ,\ )_Q$.

Introduction

Quantum antibrackets in physical literature has been introduced by Batalin and Marnelius (I.A. Batalin, R. Marnelius, Phys.Lett.B. (1998)) and in mathematical literature known as derived brackets (Yv. Kosmann-Schwarzbach, Lett.Math.Phys. (2004); Th. Voronov, J.Pure Appl. Algebra (2005); A.S. Cattaneo, F. Schatz, J.Pure Appl. Algebra (2008)).
Quantum 2-antibracket

$$(X,Y)_Q = \frac{1}{2}\big([X,[Q,Y]] - [Y,[Q,X]](-1)^{(\varepsilon(X)+1))(\varepsilon(Y)+1))}\big),$$
$$\frac{1}{2}[Q,Q] = Q^2 = 0, \quad [X,Y] = XY - YX(-1)^{\varepsilon(X)\varepsilon(Y)}.$$

General quantum antibrackets (I.A. Batalin, R. Marnelius, Theor.Math.Phys. (1999))

$$Q^2 \neq 0, \quad [Q^2, Q^2] = 0.$$

Basic properties of quantum antibrackets:
Antisymmetry

$$(X,Y)_Q = -(Y,X)_Q(-1)^{(\varepsilon(X)+1)(\varepsilon(Y)+1)}$$

Leibniz rule

$$(XY,Z)_Q - X(Y,Z)_Q - (X,Z)_Q Y(-1)^{\varepsilon(Y)(\varepsilon(Z)+1)} =$$
$$= \frac{1}{2}\big([X,Z][Y,Q](-1)^{\varepsilon(Z)(\varepsilon(Y)+1)} + [X,Q][Y,Z](-1)^{\varepsilon(Y)}\big)$$

Jacobi identity

$$(X,(Y,Z)_Q)_Q(-1)^{(\varepsilon(X)+1)(\varepsilon(Z)+1)} + \mathrm{cycle}(X,Y,Z) =$$
$$= \frac{1}{2}[(X,Y,Z)_Q,Q](-1)^{(\varepsilon(X)+1)(\varepsilon(Z)+1)}$$
$$(X,Y,Z)_Q = -\frac{1}{3}\big([X,(Y,Z)_Q](-1)^{\varepsilon(X)(\varepsilon(Z)+1)+\varepsilon(Y)} +$$
$$+\mathrm{cycle}(XYZ)\big)(-1)^{(\varepsilon(X)+1))(\varepsilon(Z)+1)}$$

Relations with antibrackets:
Let $X = X(\Phi, \Phi^*)$, $Y = Y(\Phi, \Phi^*)$ be functions of the field-antifield variables and $Q = -\Delta$, where Δ is odd Laplacian of the BV method. Then

$$[\Delta, Y] = (\Delta Y) + \mathrm{ad}(Y)(-1)^{\varepsilon(Y)}, \quad \mathrm{ad}(Y)X = (Y, X)$$

$$(X, Y)_Q = (X, Y)$$

Known applications of quantum antibrackets in theoretical physics:
BRST-invariant constraints (I.A. Batalin, I.V. Tyutin, Theor.Math.Phys. (2004))

$$[T_\alpha, T_\beta] = U^\gamma_{\alpha\beta} T_\gamma$$

$$\mathcal{T}_\alpha = (i\hbar)^{-1}[Q, \bar{\mathcal{P}}_\alpha], \quad Q^2 = 0, \quad [C^\alpha, \bar{\mathcal{P}}_\beta] = i\hbar\delta^\alpha_\beta$$

$$Q = C^\alpha T_\alpha + \frac{1}{2} C^\beta C^\alpha U^\gamma_{\alpha\beta} \bar{\mathcal{P}}_\gamma + \cdots , \quad \mathcal{T}_\alpha = T_\alpha + C^\beta U^\gamma_{\alpha\beta} \bar{\mathcal{P}}_\gamma + \cdots$$

$$[Q, \mathcal{T}_\alpha] = 0, \qquad (\mathcal{T}_\alpha, \mathcal{T}_\beta)_Q = 0$$

In group and quasigroup cases

$$Q = C^\alpha T_\alpha + \frac{1}{2} C^\beta C^\alpha U^\gamma_{\alpha\beta} \bar{\mathcal{P}}_\gamma, \quad \mathcal{T}_\alpha = T_\alpha + C^\beta U^\gamma_{\alpha\beta} \bar{\mathcal{P}}_\gamma$$

Representation of vector fields for arbitrary gauge algebra (I.A. Batalin, PML, Phys.Lett.B (2015))

$$A_\mu(x) = [\mathcal{A}_\mu(x), Q], \quad Q^2 = 0, \quad [A_\mu(x), Q] = 0$$

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = [Q, \mathcal{G}_{\mu\nu}]$$

$$\mathcal{G}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + (\mathcal{A}_\mu, \mathcal{A}_\nu)_Q$$

$$\mathcal{L} = -\frac{1}{2}\mathrm{tr}\big([Q, \mathcal{G}_{\mu\nu}][Q, \mathcal{G}_{\mu\nu}]\big)$$

$$\delta \mathcal{G}_{\mu\nu} = (\mathcal{G}_{\mu\nu}, \Lambda)_Q$$

Introduction

Heisenberg equation for superfield representation of BV-formalism (I.A. Batalin, PML, Eur.Phys.J.C. (2017))

Superfield Schroedinger equation

$$i\hbar D\Psi = Q\Psi, \quad D = \partial_\tau + \tau\partial_t, \quad Q = \Delta - F, \quad \Psi = \Psi(t,\tau,Z)$$
$$F = F(Z,P), \quad \Gamma = (Z^A, P_A), \quad [Z^A, P_B] = i\hbar\delta^A_B, \quad Z = (\Phi, \Phi^*)$$

$$i\hbar\partial_t\Psi = \mathcal{H}\Psi, \quad \mathcal{H} = (i\hbar)^{-1}\big([\Delta, F] - \frac{1}{2}[F,F]\big), \quad [\mathcal{H}, Q] = 0$$
$$\Psi(t,\tau,Z) = (1 + \tau(i\hbar)^{-1}Q)\Psi_0(t,Z)$$
$$[\Delta,\mathcal{H}] = 0 \;\rightarrow\; \Delta\Psi_0 = 0, \quad \Psi_0 = \exp\{\frac{i}{\hbar}W\}$$

Superfield Heisenberg equation

$$i\hbar D\tilde{\Gamma} = [\tilde{Q}, \tilde{\Gamma}], \quad \tilde{\Gamma} = \tilde{\Gamma}(t,\tau)$$
$$(i\hbar)^2\partial_t\tilde{\Gamma} = -\frac{2}{3}(\tilde{\Gamma}, \tilde{Q})_{\tilde{Q}}$$

Basics on non-polarized quantum antibrackets

Let Q be a Fermion nilpotent operator,

$$Q : \varepsilon(Q) = 1, \quad Q^2 = 0,$$

and let B be an arbitrary Boson operator,

$$B : \varepsilon(B) = 0.$$

A quantum 2-antibracket,

$$(B, B)_Q =: -[B, [B, Q]].$$

A quantum 3-antibracket

$$(B, B, B)_Q =: -[B, [B, [B, Q]],$$

and so on. Non-polarized form of quantum antibrackets has very simple representation. The main property

$$[Q, (B, B)_Q] = [[Q, B], [Q, B]].$$

Generating operator for non-polarized quantum antibrackets

$$\tilde{Q} = UQU^{-1}, \quad U = \exp\{\lambda B\}, \quad \tilde{Q}^2 = 0, \quad \tilde{Q}|_{\lambda=0} = Q$$

$$\partial_\lambda \tilde{Q} = [B, \tilde{Q}]$$

$$\partial_\lambda^2 \tilde{Q} = [B, [B, \tilde{Q}]], \quad \partial_\lambda^2 \tilde{Q}|_{\lambda=0} = -(B, B)_Q$$

$$\partial_\lambda^3 \tilde{Q} = [B, [B, [B, \tilde{Q}]], \quad \partial_\lambda^3 \tilde{Q}|_{\lambda=0} = -(B, B, B)_Q$$

and so on.

The non-polarized Jacobi identity reads

$$6(B,(B,B)_Q)_Q = [(B,B,B)_Q, Q],$$

as an identity with respect to B.
The main property ($Q^2 = 0$)

$$[Q,(B,B)_Q] = [[Q,B],[Q,B]].$$

All higher non-polarized identities follow from two basic relations

$$6(B,(B,B)_{\tilde{Q}})_{\tilde{Q}} = [(B,B,B)_{\tilde{Q}}, \tilde{Q}],$$

$$[\tilde{Q},(B,B)_{\tilde{Q}}] = [[\tilde{Q},B],[\tilde{Q},B]]$$

by differentiating with respect to λ and putting then $\lambda = 0$.

Indeed, from the nilpotency of $\tilde{Q}$,

$$\tilde{Q}^2 = 0$$

we have

$$\partial_\lambda \tilde{Q}^2 = [[B, \tilde{Q}], \tilde{Q}] = 0$$

$$\partial_\lambda^2 \tilde{Q}^2 = -[(B, B)_{\tilde{Q}}, \tilde{Q}] + [[B, \tilde{Q}], [B, \tilde{Q}]] = 0$$

$$\partial_\lambda^3 \tilde{Q}^2 = \frac{1}{2}\big([(B, B, B)_{\tilde{Q}}, \tilde{Q}] - 6(B, (B, B)_{\tilde{Q}})_{\tilde{Q}}\big) = 0,$$

$$\begin{aligned}\partial_\lambda^4 \tilde{Q}^2 \quad = \quad & -2\big((B, (B, B, B,))_{\tilde{Q}} + \frac{1}{4}[(B.B, B, B)_{\tilde{Q}}, \tilde{Q}] - \\ & -3[(B, B)_{\tilde{Q}}, (B, B)_{\tilde{Q}}]\big) = 0\end{aligned}$$

and so on.

Polarization

Principal question: Is non-polarized form of quantum antibracket algebra sufficient to reproduce all polarized relations? The answer is "Yes".
Let the Boson B be of the form

$$B = \alpha X + \beta Y + \gamma Z, \quad (\varepsilon(X) = \varepsilon(\alpha) \text{ and so on})$$

Then the polarized quantum 2- and 3-antibrackets

$$\partial_\alpha \partial_\beta \frac{1}{2}(B, B)_Q (-1)^{\varepsilon(\beta)} = (X, Y)_Q.$$

$$\partial_\alpha \partial_\beta \partial_\gamma \frac{1}{6}(B, B, B)_Q (-1)^{\varepsilon(\beta)} = (X, Y, Z)_Q.$$

Polarization

The polarized version of the Jacobi relation

$$
\begin{aligned}
&(X,(Y,Z)_Q)_Q(-1)^{(\varepsilon_X+1)(\varepsilon_Z+1)} + \mathrm{cycle}(X,Y,Z) = \\
&= \partial_\alpha\partial_\beta\partial_\gamma(B, \frac{1}{2}(B,B)_Q)_Q(-1)^{(\varepsilon_X+1)(\varepsilon_Z+1)+\varepsilon_Y} = \\
&= \partial_\alpha\partial_\beta\partial_\gamma \frac{1}{2}\frac{1}{6}\,[(B,B,B)_Q,Q](-1)^{(\varepsilon_X+1)(\varepsilon_Z+1)+\varepsilon_Y} = \\
&= \frac{1}{2}[(X,Y,Z)_Q,Q](-1)^{(\varepsilon_X+1)(\varepsilon_Z+1)}.
\end{aligned}
$$

The generating equations for the quantum antibracket algebra.
Let us introduce an operator valued exponential

$$U = \exp\{\lambda^a f_a\}, \quad U|_{\lambda=0} = 1,$$

where $\{f_a, a = 1, 2, ...\}$, is a chain of operators, $\varepsilon(f_a) = \varepsilon_a$, and λ^a are parameters, $\varepsilon(\lambda^a) = \varepsilon_a$.
Introduce the U-transformed Q-operator,

$$\tilde{Q} = UQU^{-1}, \quad \tilde{Q}^2 = 0.$$

Omitting details of calculations one can proof that the higher λ derivatives of $\tilde{Q}$ do yield all higher quantum antibrackets,

$$-(\partial_{a_1} \cdots \partial_{a_n} \tilde{Q})(-1)^{E_n}|_{\lambda=0} = (f_{a_1}, ..., f_{a_n})_Q,$$

$$E_n = \sum_{k=1}^{[n/2]} \varepsilon_{a_{2k}},$$

Conclusion

- We have studied non-polarized form of quantum antibracket algebra and demonstrated their simple representation.
- We have constructed the generating operator for non-polarized quantum antibracket algebra. It reproduces all relations existing between quantum antibrackets.
- We have proved that polarized quantum antibracket algebra can be restored by applying polarization and parametrization procedures.

THANK YOU
FOR ATTENTION!