Unitary quantization and para-Fermi statistics

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In the papers by A.B. Govorkov (1979) and independently by T.D. Palev (1977) a formalism for the quantization of fields based on the relations of the Lie algebra of the unitary group $SU(2M+1)$ was set up. The suggested scheme of quantization was called "the uniquantization". In this work we would like to investigate in more detail some properties of the relations obtained by Govorkov, and in particular, to establish a connection between the unitary quantization and para-Fermi statistics of order 2.

Let $(a_k, a_k^\dagger)$ and $(b_k, b_k^\dagger)$ be two sets of the annihilation and creation operators obeyed the Green commutation relations (H.S. Green (1953)):

$$[[\hat{a}_k, \hat{a}_l], \hat{a}_m] = 2\hat{\delta}_{lm}\hat{a}_k - 2\hat{\delta}_{km}\hat{a}_l, \tag{1}$$

$$[[\hat{b}_k, \hat{b}_l], \hat{b}_m] = 2\hat{\delta}_{lm}\hat{b}_k - 2\hat{\delta}_{km}\hat{b}_l, \tag{2}$$

where $k, l, m = 1, 2, \ldots, M$. The operator with hat above $\hat{a}_k$ stands for a_k or $a_k^\dagger$ and $\hat{\delta}_{kl} = \delta_{kl}$ when $\hat{a}_k = a_k(a_k^\dagger)$ and $\hat{a}_l = a_l^\dagger(a_l)$, and $\hat{\delta}_{kl} = 0$ otherwise. The same is also valid for the operator $\hat{b}_k$.

In addition to (1) there exist the mutual commutation relations of two types between the operators $\hat{a}_k$ and $\hat{b}_k$:

1. trilinear relations

$$[[\hat{b}_m, \hat{a}_k], \hat{a}_l] = 4\hat{\delta}_{km}\hat{b}_l + 2\hat{\delta}_{lk}\hat{b}_m + 2\hat{\delta}_{lm}\hat{b}_k, \tag{3}$$

$$[[\hat{a}_m, \hat{b}_k], \hat{b}_l] = 4\hat{\delta}_{km}\hat{a}_l + 2\hat{\delta}_{lk}\hat{a}_m + 2\hat{\delta}_{lm}\hat{a}_k, \tag{4}$$

2. bilinear relations

$$[\hat{a}_k, \hat{b}_m] = [\hat{a}_m, \hat{b}_k], \qquad [\hat{a}_k, \hat{a}_m] = [\hat{b}_k, \hat{b}_m]. \tag{5}$$

As is well known, the commutation relations (1) and (2) generate an algebra which is isomorphic to the algebra of the *orthogonal group* $SO(2M+1)$. The other relations (3) – (5) complete this algebra to the algebra of the unitary group $SU(2M+1)$. The particle-number operator

$$N = \frac{1}{2}\sum_{k=1}^{M}\left([a_k^{\dagger}, a_k] + p\right)\left(\equiv \frac{1}{2}\sum_{k=1}^{M}\left([b_k^{\dagger}, b_k] + p\right)\right) \tag{6}$$

together with the algebra (1) – (5) fixes the unitary quantization scheme.

Each field operator is expanded into the so-called Green components:

$$a_k = \sum_{\alpha=1}^{p} a_k^{(\alpha)}, \quad b_m = \sum_{\alpha=1}^{p} b_m^{(\alpha)}, \tag{7}$$

where p is the order of parastatistics. Each pair of components belonging to the same field satisfies the commutation rules

$$\begin{aligned} \{a_k^{(\alpha)}, a_l^{\dagger(\alpha)}\} &= \delta_{kl}, \quad \{a_k^{(\alpha)}, a_l^{(\alpha)}\} = 0, \\ [a_k^{(\alpha)}, a_l^{(\beta)}] &= 0, \quad [a_k^{(\alpha)}, a_l^{\dagger(\beta)}] = 0, \quad \alpha \neq \beta, \end{aligned} \tag{8}$$

and similarly for the ϕ_b field. We *postulate* that components $a_k^{(\alpha)}$, $b_m^{(\beta)}$ and an additional operator Ω satisfy the following system of commutation rules:

$$[b_m^{(\alpha)}, a_k^{\dagger(\alpha)}] = 2\delta_{mk}\Omega, \quad [a_k^{(\alpha)}, b_m^{\dagger(\alpha)}] = 2\delta_{mk}\Omega^{\dagger}, \tag{9}$$

$$[a_k^{(\alpha)}, b_m^{(\alpha)}] = [a_k^{\dagger(\alpha)}, b_m^{\dagger(\alpha)}] = 0, \tag{10}$$

$$[\Omega, a_k^{(\alpha)}] = b_k^{(\alpha)}, \quad [\Omega, b_m^{(\alpha)}] = -a_m^{(\alpha)}, \qquad \alpha \neq \beta, \tag{11}$$

$$\{a_k^{(\alpha)}, b_m^{(\beta)}\} = \{a_k^{\dagger(\alpha)}, b_m^{(\beta)}\} = \{a_k^{(\alpha)}, b_m^{\dagger(\beta)}\} = \{a_k^{\dagger(\alpha)}, b_m^{\dagger(\beta)}\} = 0. \tag{12}$$

It can easily be shown that the fields obeying the rules (9)–(12) verify the set of the bilinear relations (5) for any order p of parastatistics. Only one of them requires special consideration, namely,

$$[a_k^\dagger, b_m] = [a_m, b_k^\dagger]. \tag{13}$$

By using the relations (9) and (11), we find for the left-hand side of (13):

$$[a_k^\dagger, b_m] = -2p\delta_{km}(\Omega + \Omega^\dagger) + [a_m, b_k^\dagger].$$

Thus, the bilinear relation (13) will hold if the operator Ω satisfies the following additional condition:

$$\Omega + \Omega^\dagger = 0. \tag{14}$$

Further we consider the trilinear relations for two different parafields. A particular consequence of the general formula (3) is the following three relations

$$[[b_m, a_k^\dagger], a_l] = 2\delta_{kl} b_m + 4\delta_{km} b_l, \tag{15}$$

$$[[a_l, b_m], a_k^\dagger] = -2\delta_{kl} b_m - 2\delta_{km} b_l, \tag{16}$$

$$[[a_k^\dagger, a_l], b_m] = -2\delta_{km} b_l. \tag{17}$$

Let us consider the trilinear relation (15). Taking into account (8), (12) and the generalized Jacoby's identities

$$[A,[B,C]] = -[B,[C,A]] - [C,[A,B]], \tag{18}$$

$$[A,[B,C]] = \{C,\{A,B\}\} - \{B,\{A,C\}\}, \tag{19}$$

$$\{A,[B,C]\} = \{B,[C,A]\} - [C,\{A,B\}], \tag{20}$$

$$[A,\{B,C\}] = -[B,\{C,A\}] - [C,\{A,B\}], \tag{21}$$

we finally obtain, instead of the left-hand side of (15)

$$[[b_m, a_k^\dagger], a_l] = 2(p-1)\delta_{lk} b_m + 2p\delta_{mk} b_l + \sum_{\alpha \neq \beta \neq \gamma} \{b_m^{(\alpha)}, \{a_l^{(\gamma)}, a_k^{\dagger(\beta)}\}\}. \tag{22}$$

We see that this expression reproduces (15) in the only case, when $p = 2$. In this special case the last term on the right-hand side of (22) is simply absent and the numerical coefficients in the other terms take correct values.

The same reasoning for the l.h.s. of the relation (16) leads to

$$[[a_l, b_m], a_k^\dagger] = -2(p-1)\delta_{kl} b_m - 2(p-1)\delta_{mk} b_l - \sum \{b_m^{(\beta)}, \{a_k^{\dagger(\gamma)}, a_l^{(\alpha)}\}\}.$$

We see again that this expression reproduces (16) only for the case $p = 2$.

Now our task is to derive the commutation rules involving the para-Grassmann numbers ξ_k and operators a_k, b_m. In the case of a single para-Fermi field, for instance ϕ_a, such commutation rules were suggested in the paper by M. Omote and S. Kamefuchi (1979):

$$[a_k,[a_l,\xi_m]]=0, \quad [a_k,[a_l^\dagger,\xi_m]]=2\delta_{kl}\xi_m,$$
$$[\xi_k,[\xi_l,a_m]]=0, \quad [\xi_k,[\xi_l,\xi_m]]=0. \qquad (23)$$

For para-Grassmann numbers ξ_k the Green representation is also true

$$\xi_k=\sum_{\alpha=1}^{p}\xi_k^{(\alpha)}.$$

The bilinear commutation relations for Green components $a_k^{(\alpha)}$, $b_m^{(\alpha)}$, $\xi_l^{(\alpha)}$ are:

$$\{a_k^{(\alpha)},\xi_l^{(\alpha)}\}=0, \quad \{b_m^{(\alpha)},\xi_l^{(\alpha)}\}=0, \quad \{\xi_k^{(\alpha)},\xi_l^{(\alpha)}\}=0,$$
$$[a_k^{(\alpha)},\xi_l^{(\beta)}]=0, \quad [b_m^{(\alpha)},\xi_l^{(\beta)}]=0, \quad [\xi_k^{(\alpha)},\xi_l^{(\beta)}]=0, \quad \alpha\neq\beta \qquad (24)$$

plus Hermitian conjugation. They turn (23) into identity.

A direct consequence of the commutation rules (9)−(12) and (24) for the Green components is the following trilinear relations:

$$\{b_m, [a_k^\dagger, \xi_l]\} = 2\delta_{mk}\{\xi_l, \Omega\}, \quad \{a_k, [b_m^\dagger, \xi_l]\} = -2\delta_{mk}\{\xi_l . \Omega\}. \tag{25}$$

Here we can make a step forward and *postulate* the following relation:

$$\{\xi_l, \Omega\} = \Lambda \xi_l, \tag{26}$$

where Λ is some constant satisfying the condition: $\Lambda = -\Lambda^*$. Thus, instead of (25), now we have

$$\{b_m, [a_k^\dagger, \xi_l]\} = 2\Lambda\delta_{mk}\xi_l, \qquad \{a_k, [b_m^\dagger, \xi_l]\} = 2\Lambda^*\delta_{mk}\xi_l. \tag{27}$$

It is these relations that we accept as the definitions of the trilinear relations involving two different parafields ϕ_a, ϕ_b and para-Grassmann numbers ξ_l.

Further, A.B. Govorkov (1979) has introduced an important operator $\widetilde{N}$:

$$\widetilde{N} = \frac{i}{2M+1}\sum_{k=1}^{M}\left([a_k^\dagger, b_k] + [b_k^\dagger, a_k] + \lambda\right), \tag{28}$$

where λ is some real constant different from zero. The operator $\widetilde{N}$ posses-ses the following properties:

$$[i\widetilde{N}, a_k] = b_k, \qquad [i\widetilde{N}, b_k] = -a_k. \tag{29}$$

A connection between the operators $i\tilde{N}$ and Ω

There exists a certain connection between the operators $i\tilde{N}$ and Ω, namely

$$i\widetilde{N} = \frac{2M}{2M+1}\,\Omega - \frac{1}{2(2M+1)}\sum_{k=1}^{M}\left([a_k^{\dagger(1)}, b_k^{(2)}] + [a_k^{\dagger(2)}, b_k^{(1)}]\right) - \frac{M}{2(2M+1)}\,\lambda.$$

Taking into account (26) and (27) from the last expression it follows that

$$\{\xi_l, i\widetilde{N}\} = \tilde{\Lambda}\xi_l, \quad \tilde{\Lambda} = \frac{M}{2M+1}\,(2\Lambda - \lambda). \tag{30}$$

Further, by using the relations (11), we get for the Green components $a_k^{(\alpha)}$:

$$[i\widetilde{N}, a_k^{(1)}] = \frac{1}{2M+1}\left(2M b_k^{(1)} + b_k^{(2)}\right), \; [i\widetilde{N}, a_k^{(2)}] = \frac{1}{2M+1}\left(2M b_k^{(2)} + b_k^{(1)}\right).$$

Similar calculations for commutators with Green components $b_m^{(\alpha)}$, lead us to

$$[i\widetilde{N}, b_m^{(1)}] = \frac{(-1)}{2M+1}\left(2M a_m^{(1)} + a_m^{(2)}\right), \; [i\widetilde{N}, b_m^{(2)}] = \frac{(-1)}{2M+1}\left(2M a_m^{(2)} + a_m^{(1)}\right).$$

In spite of somewhat unusual form of these commutation rules, they correctly reproduce the relations (29) when $a_k = a_k^{(1)} + a_k^{(2)}$, $b_m = b_m^{(1)} + b_m^{(2)}$.

Commutation relations with the operator $\mathrm{e}^{\alpha i \widetilde{N}}$

The basic relations determining a rule of rearrangement between the operator $\mathrm{e}^{\alpha i \widetilde{N}}$ and a_k, b_m have the following form

$$\mathrm{e}^{\alpha i \widetilde{N}} a_k = (a_k \cos\alpha + b_k \sin\alpha)\mathrm{e}^{\alpha i \widetilde{N}},$$

$$\mathrm{e}^{\alpha i \widetilde{N}} b_m = (b_m \cos\alpha - a_m \sin\alpha)\mathrm{e}^{\alpha i \widetilde{N}},$$

where α is an arbitrary real number. We are interested in two particular cases

1. for $\alpha = \pm\pi$ we have

$$\{\mathrm{e}^{\pm\pi i \widetilde{N}}, a_k\} = 0, \quad \{\mathrm{e}^{\pm\pi i \widetilde{N}}, b_m\} = 0; \tag{31}$$

2. for $\alpha = \pm\pi/2$ we have

$$\mathrm{e}^{\pm\pi/2 i \widetilde{N}} a_k = \pm b_k \mathrm{e}^{\pm\pi/2 i \widetilde{N}}, \quad \mathrm{e}^{\pm\pi/2 i \widetilde{N}} b_m = \mp a_m \mathrm{e}^{\pm\pi/2 i \widetilde{N}}. \tag{32}$$

The relations (32) define two equivalent "mapping" the operator b_k into the operator a_k (and vice versa):

$$a_k = \pm \mathrm{e}^{\mp\pi/2 i \widetilde{N}} b_k \mathrm{e}^{\pm\pi/2 i \widetilde{N}}, \quad a_k = \mp \mathrm{e}^{\pm\pi/2 i \widetilde{N}} b_k \mathrm{e}^{\mp\pi/2 i \widetilde{N}}. \tag{33}$$

Further, a rule of rearrangement between the operator $\mathrm{e}^{\alpha i \widetilde{N}}$ and the para-Grassmann numbers ξ_k has the form:

$$\mathrm{e}^{\alpha i \widetilde{N}} \xi_k = \xi_k \mathrm{e}^{\alpha \tilde{\Lambda}} \mathrm{e}^{-\alpha i \widetilde{N}}. \tag{34}$$

Let us consider the mapping of the trilinear relation including the operator a_m once. Making use of (33) and (34), we result in the following expression:

$$[\xi_k, [\xi_l, a_m]] = 0 \quad \Rightarrow \quad \{\xi_k, \{\xi_l, b_m\}\} = 0. \tag{35}$$

Further, the mapping of more nontrivial trilinear relation leads to

$$\{a_k, [\xi_l, b_m^\dagger]\} = 2\Lambda\delta_{mk}\xi_l \quad \Rightarrow \quad [b_k, \{\xi_l, a_m^\dagger\}] = 2\Lambda\delta_{mk}\xi_l. \tag{36}$$

A mapping of the trilinear relation containing the operators a_k and $a_l^\dagger$ simultaneously is

$$[a_k, [a_l^\dagger, \xi_m]] = 2\delta_{kl}\xi_m. \quad \Rightarrow \quad \{b_k, \{b_l^\dagger, \xi_m\}\} = 2\delta_{kl}\xi_m.$$

Contrary to the expectation, under the mapping (33) the relations do not turn into similar relations with the only replacement $a_m \to b_m$. We see that in addition to the replacement, all commutators are replaced by anticommutators and vice versa. This circumstance can take place exceptionally for parastatistics of order 2.

The peculiarity of all examples considered above is that the para-Grassmann number ξ_k always enters into the commutator or anticommutator along with the operator a_k or b_m (or with their Hermitian conjugation).

Action of the operators Ω and $i\widetilde{N}$ on the vacuum state

We consider the problem of acting the operators Ω and $\widetilde{N}$ on the vacuum state $|0\rangle$. For the operator $\widetilde{N}$, Eq. (28), we have

$$\widetilde{N}|0\rangle = -\frac{i}{2(2M+1)} \sum_{k=1}^{M} b_k a_k^{\dagger}|0\rangle + \frac{iM}{2(2M+1)} \lambda|0\rangle. \tag{37}$$

If one uses an additional condition of a unique vacuum state $|0\rangle$:

$$b_m a_k^{\dagger}|0\rangle = 0, \qquad a_k b_m^{\dagger}|0\rangle = 0, \tag{38}$$

obtained by O. Greenberg, A. Messiah (1965) within their extension of Green trilinear relations (1) and (2) to the case of different parafields, then we get

$$\widetilde{N}|0\rangle = \lambda \frac{iM}{2(2M+1)} |0\rangle.$$

Hence, if we wanted to demand the fulfillment of the condition

$$\widetilde{N}|0\rangle = 0 \tag{39}$$

by analogy with similar condition for the particle-number operator: $N|0\rangle = 0$, we would lead to the trivial requirement: $\lambda = 0$. The latter actually results in the degeneration of the theory under consideration. The only way to avoid this is to give up the condition (38).

Action of the operators Ω and $i\widetilde{N}$ on the vacuum state

We have shown that within the framework of the unitary quantization we come to the following additional conditions of a unique vacuum state $|0\rangle$, instead of (38):

$$b_m a_k^\dagger|0\rangle = c\,\delta_{mk}|0\rangle, \qquad a_k b_m^\dagger|0\rangle = c^*\delta_{mk}|0\rangle, \tag{40}$$

where c is some, generally speaking, complex constant, different from zero. In this case, acting the operator $\widetilde{N}$ on the vacuum is

$$\widetilde{N}|0\rangle = -(c-\lambda)\,\frac{iM}{2(2M+1)}\,|0\rangle.$$

The requirement of the fulfillment of three conditions

$$\widetilde{N}|0\rangle = 0, \qquad \{\xi_l,\Omega\} = \Lambda\xi_l, \qquad \{\xi_l, i\widetilde{N}\} = \frac{M}{2M+1}\,(2\Lambda-\lambda)\xi_l$$

results in a rule of acting the operator Ω on the vacuum:

$$\Omega|0\rangle = \frac{1}{4}\,c|0\rangle$$

and in an unique fixing of the constants Λ and c in terms of the parameter λ:

$$c = \lambda, \qquad \Lambda = \frac{1}{2}\,\lambda. \tag{41}$$

It is the only parameter which remains undefined in the theory in question.

M. Omote and S. Kamefuchi (1979) have introduced the coherent state of a system of para-Fermi oscillators a_k in the following form:

$$|(\xi)_p\,;a\rangle = \exp\Big(-\frac{1}{2}\sum_{l=1}^{M}[\xi_l, a_l^\dagger]\Big)|0\rangle, \qquad (42)$$

in so doing

$$a_k|(\xi)_p\,;a\rangle = \xi_k|(\xi)_p\,;a\rangle. \qquad (43)$$

In notation of the coherent state $|(\xi)_p\rangle$ accepted by these authors, we have inserted additional symbol a to emphasize that this state is associated with the field ϕ_a. In a similar way, we can define a coherent state for a system of para-Fermi oscillators b_k.

In the general case the coherent state for the b-operators will never be the coherent state for the a-operators. However, for parastatistics of order 2, within the framework of uniquantization the situation is somewhat different:

$$a_k|(\xi)_2\,;b\rangle = \Lambda\xi_k\Big(\sum_s \frac{1}{(s+1)!}\,[\xi_m, b_m^\dagger]^s\Big)|(\xi)_2\,;b\rangle. \qquad (44)$$

The complexity of expression on r.h.s. of (44) is ultimately a consequence of *"involving"* the coherent state with the opposite sign of the variable ξ_k.

Actually, if we act by the operator $[\xi_l, b_l^\dagger]$ on (44), we will have

$$a_k [\xi_l, b_l^\dagger]\,|(\xi)_2\,;b\rangle = \Lambda \xi_k \Big(|(\xi)_2\,;b\rangle - |(-\xi)_2\,;b\rangle \Big). \qquad (45)$$

The state $|(-\xi)_2\,;b\rangle$ in turn can be presented as a result of acting on the initial coherent state $|(\xi)_2;b\rangle$ by the parafermion number counter $(-1)^N$ ("G-parity operator"), with the particle-number operator (6), i.e.

$$(-1)^N |(\xi)_2\,;b\rangle = |(-\xi)_2\,;b\rangle.$$

We might observe in passing that the relation of the following form

$$[a_k, [b_l^\dagger, \xi_l]]|(\xi)_2\,;b\rangle = 2\Lambda \xi_k (-1)^N |(\xi)_2\,;b\rangle \qquad (46)$$

will be a consequence of (45) and of the basic relation

$$\{a_k, [b_m^\dagger, \xi_l]\} = 2\Lambda^* \delta_{mk} \xi_l.$$

The relation (46) may, in turn, implies

$$[a_k, [b_m^\dagger, \xi_l]] = 2\Lambda \delta_{mk} \xi_l (-1)^N.$$

Within the framework of the usual Fermi statistics the state:

$$(-1)^N|\xi\rangle = |(-\xi)\rangle$$

was considered by E. D'Hoker and D.G. Gagné (1996) in the context of the construction of worldline path integral for the *imaginary* part of the effective action. It is also interesting to note that the number counter enters into the so-called deformed Heisenberg algebra (the Calogero-Vasiliev oscillator) (M.A. Vasiliev, 1990, 1991) involving the reflection operator $R = (-1)^N$ and a deformation parameter $\nu \in \mathbb{R}$.

Further, M. Plyushchay (1997) has shown that the single-mode deformed Heisenberg algebra with reflection has finite-dimensional representations of some deformed parafermion algebra which at $\nu = -3$ is reduced to the standard parafermionic algebra of order 2. This suggests that there is a certain connection between the unitary quantization and the deformed Heisenberg algebra.

Here, we should also like to consider a problem of mapping the expression (43). As the transformation connecting the operators a_k and b_k we take the relation:

$$a_k = -\,\mathrm{e}^{\pi/2 i\widetilde{N}} b_k \,\mathrm{e}^{-\pi/2 i\widetilde{N}}. \tag{47}$$

The calculations are somewhat lengthly and eventually lead to the simple answer:

$$a_k\,\mathrm{e}^{-\frac{1}{2}[\xi_l,\, a_l^{\dagger}]}|0\rangle = \xi_k\,\mathrm{e}^{-\frac{1}{2}[\xi_l,\, a_l^{\dagger}]}|0\rangle \;\Rightarrow\; b_k\,\mathrm{e}^{\frac{1}{2}\{\xi_l,\, b_l^{\dagger}\}}|0\rangle = \xi_k\,\mathrm{e}^{\frac{1}{2}\{\xi_l,\, b_l^{\dagger}\}}|0\rangle. \tag{48}$$

The first expression in (48) is true by virtue of the the trilinear relations

$$[\xi_k, [\xi_l, a_m]] = 0 \quad \text{and} \quad [a_k, [a_l^{\dagger}, \xi_m]] = 2\delta_{kl}\xi_m, \tag{49}$$

while the second one holds by virtue of *dual* representation of (49) under mapping (47)

$$\{\xi_k, \{\xi_l, b_m\}\} = 0 \quad \text{and} \quad \{b_k, \{b_l^{\dagger}, \xi_m\}\} = 2\delta_{kl}\xi_m. \tag{50}$$

A consequence of Eqs. (49) and (50) is an existence of two alternative definitions of the parafermion coherent state:

$$|(\xi)_2\rangle = \exp\Big(-\frac{1}{2}\sum\nolimits_l [\xi_l, a_l^{\dagger}]\Big)|0\rangle \quad \text{and} \quad |(\xi)_2\rangle = \exp\Big(\frac{1}{2}\sum\nolimits_l \{\xi_l, a_l^{\dagger}\}\Big)|0\rangle.$$

We stress that an existence of these two definitions is inherent in the para-Fermi statistics of order 2 itself and is not specific to the unitary quantization scheme.

In both cases the main property of the coherent state is fulfilled

$$a_k|(\xi)_2\rangle = \xi_k|(\xi)_2\rangle$$

and, besides, the *overlap* function has its usual form

$$\langle(\bar{\xi}')_2|(\xi)_2\rangle = \mathrm{e}^{\frac{1}{2}[\bar{\xi}'_l,\, \xi_l]}.$$

The precise meaning of appearance of such "twins" remains unclear for us. Perhaps one reason of a purely algebraic nature is the fact that of the four basic identities (18)–(21), only two ones are independent, namely (19) and (20). This circumstance and its consequence were analyzed in detail in paper by P.M. Lavrov, O.V. Radchenko, and I.V. Tyutin (2014). In particular, the Jacobi identity

$$[A, [B, C]] = -[B, [C, A]] - [C, [A, B]]$$

is a consequence of the generalized identity

$$[A, [B, C]] = \{C, \{A, B\}\} - \{B, \{A, C\}\}.$$

The latter contains double anticommutators on r.h.d. side as in (50). This hints that one of the relations (49) and (50) is a consequence of the other for $p = 2$. In any case we may state that the para-Fermi statistics of order 2 is a very special case of parastatistics as well as Fermi statistics of order 1.

The Klein transformation allows to lead the initial relations for the Green components of operators, which contain both commutators and anticommutators, to the normal commutation relations for p ordinary Fermi fields (K. Drühl, R. Haag, J.E. Roberts (1970)).

In the problem under consideration with two different para-Fermi fields ϕ_a and ϕ_b of order $p = 2$, we need at least two Klein operators, which we designate as $H_A^{(2)}$ and $H_B^{(1)}$. It is necessary to define the Klein transformation of Green's components $a_k^{(\alpha)}$ and $b_m^{(\alpha)}$ so that to reduce simultaneously to the normal form both the standard commutation relations (8) separately for each set $\{a_k^{(\alpha)}\}$, $\{b_m^{(\alpha)}\}$ and the commutation relations (9)–(12) of the mixed type. We state that the required Klein transformation has the form:

$$\begin{aligned} a_k^{(1)} &= A_k^{(1)} H_A^{(2)}, & b_m^{(1)} &= -iB_m^{(1)} H_B^{(1)}, \\ a_k^{(2)} &= iA_k^{(2)} H_A^{(2)}, & b_m^{(2)} &= B_m^{(2)} H_B^{(1)}, \end{aligned} \tag{51}$$

where $A_k^{(\alpha)}$ and $B_m^{(\alpha)}$ are new Green's components satisfying the following commutation rules with Klein operators $H_A^{(2)} = (-1)^{N_A^{(2)}}$, $H_B^{(1)} = (-1)^{N_B^{(1)}}$,

where $N_A^{(2)} \equiv (1/2) \sum_{k=1}^{M} [A_k^{\dagger(2)}, A_k^{(2)}] + M/2,$ (similarly for $N_B^{(1)}$).

$$\begin{cases}[A_k^{(1)}, H_A^{(2)}] = 0, & \{A_k^{(1)}, H_B^{(1)}\} = 0,\\ \{A_k^{(2)}, H_A^{(2)}\} = 0, & [A_k^{(2)}, H_B^{(1)}] = 0,\end{cases} \quad \begin{cases}\{B_m^{(1)}, H_B^{(1)}\} = 0, & [B_m^{(1)}, H_A^{(2)}] = 0,\\ [B_m^{(2)}, H_B^{(1)}] = 0, & \{B_m^{(2)}, H_A^{(2)}\} = 0.\end{cases}$$

At the same time the Klein operators themselves satisfy the conditions

$$\big(H_A^{(2)}\big)^2 = \big(H_B^{(1)}\big)^2 = I, \qquad [H_A^{(2)}, H_B^{(1)}] = 0. \tag{52}$$

The Klein transformation (51) leads to the normal form of the relations (8)

$$\{A_k^{(\alpha)}, A_l^{\dagger(\alpha)}\} = \delta_{kl}, \quad \{A_k^{(\alpha)}, A_l^{\dagger(\beta)}\} = 0, \quad \{A_k^{(\alpha)}, A_l^{(\beta)}\} = 0, \quad \alpha \neq \beta$$

and the same is hold for $B_m^{(\alpha)}$. Instead of (9) and (10), we now get

$$\{B_m^{(\alpha)}, A_k^{\dagger(\alpha)}\} = (2/i)\,\delta_{mk}\,\widetilde{\Omega}, \quad \{A_k^{(\alpha)}, B_m^{(\alpha)}\} = 0, \quad \{A_k^{\dagger(\alpha)}, B_m^{\dagger(\alpha)}\} = 0,$$

and the relations (11) with the use of $\Omega = H_A^{(2)}\,\widetilde{\Omega}\,H_B^{(1)}$ transform to

$$\{A_k^{(\alpha)}, \widetilde{\Omega}\} = iB_k^{(\alpha)}, \quad \{B_m^{(\alpha)}, \widetilde{\Omega}\} = iA_m^{(\alpha)}. \tag{53}$$

Finally, the anticommutation relations (12) retain the form with the replacements $a_k^{(\alpha)} \to A_k^{(\alpha)}$ and $b_m^{(\alpha)} \to B_m^{(\alpha)}$.

We would like to discuss an interesting connection between the Govorkov trilinear relations (15) – (17) and Lie-supertriple system (S. Okubo (1994)).

Let V be a vector superspace, i.e. it represents a direct sum

$$V = V_B \oplus V_F .$$

In this superspace the grade is entered by

$$\sigma(x) = \begin{cases} 0, & \text{if } x \in V_B \\ 1, & \text{if } x \in V_F \end{cases} \tag{54}$$

and the triple superproduct $[..., ..., ...]$ is defined as a trilinear mapping

$$[..., ..., ...]; \; V \otimes V \otimes V \rightarrow V.$$

The triple superproduct is subject to three conditions (we do not present their here, see S. Okubo (1994)). Besides, it is supposed that the underlying vector superspace V always possesses a bilinear form $\langle x | \, y \rangle$ satisfying

$$\begin{aligned} \langle x | \, y \rangle &= (-1)^{\sigma(x)\sigma(y)} \langle y | \, x \rangle, \\ \langle x | \, y \rangle &= 0, \;\; \text{if } \sigma(x) \neq \sigma(y). \end{aligned} \tag{55}$$

Let $P: V \to V$ be a grade-preserving linear map in V, i.e.

$$\sigma(Px) = \sigma(x), \text{ for any } x \in V$$

and we assume the validity of

$$P^2 = \lambda I, \qquad \langle x|Py\rangle = -\langle Px|y\rangle, \tag{56}$$

where I is the identity mapping in V and λ is nonzero constant. The following expression for the triple product:

$$[x, y, z] = \langle y|Pz\rangle Px - (-1)^{\sigma(x)\sigma(y)}\langle x|Pz\rangle Py - 2\langle x|Py\rangle Pz \\ + \lambda\langle y|z\rangle x - (-1)^{\sigma(x)\sigma(y)}\lambda\langle x|z\rangle y \tag{57}$$

transforms the superspace V into a Lie-supertriple system with this triple product. The Govorkov trilinear relations (15)–(17) represent particular cases of the general formula (57). In addition, the triple product contains also the standard trilinear relations for the single field ϕ_a (and ϕ_b).

Our first step is to fix two sets of operators $(a_k, a_k^\dagger)$ and $(b_k, b_k^\dagger)$, $k=\overline{1, M}$ between which we specify a map P by the rule

$$Pa_k = b_k, \quad Pb_k = -a_k \qquad \text{(similarly for } a_k^\dagger \text{ and } b_k^\dagger). \tag{58}$$

It immediately follows that: $P^2 a_k = -a_k,\ P^2 b_k = -b_k$ and thus, by virtue of the first condition in (56), the constant λ is uniquely fixed:

$$\lambda = -1. \tag{59}$$

Let us consider the second condition in (56). We set $x = a_k^\dagger$ and $y = b_m$, then, due to (58), the condition for the bilinear form $\langle \cdot | \cdot \rangle$ reduces to

$$\langle a_k^\dagger | a_m \rangle = \langle b_k^\dagger | b_m \rangle. \tag{60}$$

We fix the grade: $\sigma(a_k) = \sigma(a_k^\dagger) = 0,\ \sigma(b_m) = \sigma(b_m^\dagger) = 0$ and choose the bilinear form $\langle x | y \rangle$ to satisfy

$$\begin{aligned}
&\langle a_k^\dagger | a_m \rangle = \langle a_m | a_k^\dagger \rangle = -2\delta_{km}, && \langle b_k^\dagger | b_m \rangle = \langle b_m | b_k^\dagger \rangle = -2\delta_{km}, \\
&\langle a_k^\dagger | a_m^\dagger \rangle = \langle a_k | a_m \rangle = 0, && \langle b_k^\dagger | b_m^\dagger \rangle = \langle b_k | b_m \rangle = 0, \\
&\langle a_k^\dagger | b_m \rangle = \langle a_k | b_m \rangle = 0, && \langle b_k^\dagger | a_m \rangle = \langle b_k^\dagger | a_m^\dagger \rangle = 0.
\end{aligned} \tag{61}$$

The condition (60) is automatically satisfied. If we set $x = b_m,\ y = a_k^\dagger$, and $z = a_l$, then the triple product (57), by virtue of (58), (59), (61), gives us the relation (15). For $x = a_l,\ y = b_m,\ z = a_k^\dagger$ we reproduce (16) and so on.

One of the main reasons of appearance of the present work was a hope to develop a convenient mathematical technique, which would enable us within the framework of the Duffin-Kemmer-Petiau formalism to construct the path integral representation in parasuperspace for the inverse operator $\hat{\mathcal{L}}^{-1}(z, D)$

$$\hat{\mathcal{L}} \equiv \hat{\mathcal{L}}(z, D) = A\left(\frac{i}{\varepsilon^{1/3}(z)}\,\eta_\mu(z)\,D_\mu + mI\right), \quad \varepsilon(z) \equiv (z-q)(z-q^2),$$

where $D_\mu = \partial_\mu + ieA_\mu(x)$ is the covariant derivative and q is a primitive cubic root of unity. The operator $\hat{\mathcal{L}}(z, D)$ represents cubic root of some third order wave operator in an external electromagnetic field (Yu. Markov *et al.* (2015)). The Fock-Schwinger proper-time representation for $\hat{\mathcal{L}}^{-1}$ is

$$\frac{1}{\hat{\mathcal{L}}} \equiv \frac{\hat{\mathcal{L}}^2}{\hat{\mathcal{L}}^3} = i\int_0^\infty d\tau \int \frac{d^2\chi}{\tau^2}\,\mathrm{e}^{-i\tau\left(\hat{H}(z) - i\epsilon\right) + \frac{1}{2}\left(\tau[\chi, \hat{\mathcal{L}}] + \frac{1}{4}\tau^2[\chi, \hat{\mathcal{L}}]^2\right)},$$

where $\hat{H}(z) \equiv \hat{\mathcal{L}}^3(z, D)$, χ is a para-Grassmann variable of order $p = 2$ and as a proper para-supertime here it is necessary to take a triple (τ, χ, χ^2).

A connection with the Duffin-Kemmer-Petiau formalism

Matrix element of the operator $\hat{\mathcal{L}}^{-1}(z, D)$ in the corresponding basis of states can be considered as *the propagator of a massive vector particle in a background gauge field*. The matrices $\eta_\mu(z)$ are defined by the matrices β_μ of the Duffin-Kemmer-Petiau algebra

$$\beta_\mu \beta_\nu \beta_\lambda + \beta_\lambda \beta_\nu \beta_\mu = \delta_{\mu\nu} \beta_\lambda + \delta_{\lambda\nu} \beta_\mu$$

and by the complex deformation parameter z as follows:

$$\eta_\mu(z) \equiv \left(1 + \frac{1}{2}\, z\right) \beta_\mu - z \left(\frac{\sqrt{3}}{2}\right) \zeta_\mu, \quad \zeta_\mu = i\,[\beta_\mu, \omega], \tag{62}$$

where

$$\omega = \frac{1}{(M!)^2}\, \epsilon_{\mu_1 \mu_2 \ldots \mu_{2M}} \beta_{\mu_1} \beta_{\mu_2} \ldots \beta_{\mu_{2M}}.$$

At the end of all calculations, it should be necessary to passage to the limit $z \to q$, in particular, for the Hamilton operator we will have:

$$\hat{H} = \lim_{z \to q} \hat{H}(z) = \lim_{z \to q} \left[A \left(\frac{i}{\varepsilon^{1/3}(z)}\, \eta_\mu(z) D^\mu - mI \right) \right]^3.$$

Unfortunately, Govorkov's unitary quantization formalism has proved to be unsuitable for this purpose. Below we discuss this problem in more detail.

A connection with the Duffin-Kemmer-Petiau formalism

The easiest way to establish a connection between the DKP theory and the unitary quantization scheme is to identify literally the matrices β_μ and ζ_μ from the DKP approach with the quantities β_μ and ζ_μ that appear within uniquantization. The latter are associated with a set of operators $(a_k, a_k^\dagger)$ and $(b_k, b_k^\dagger)$ by the relations:

$$\begin{aligned} \beta_{2k-1} &= \tfrac{1}{2}\,(a_k + a_k^\dagger), \qquad & \beta_{2k} &= \tfrac{i}{2}\,(a_k - a_k^\dagger), \\ \zeta_{2k-1} &= \tfrac{1}{2}\,(b_k + b_k^\dagger), \qquad & \zeta_{2k} &= \tfrac{i}{2}\,(b_k - b_k^\dagger), \end{aligned} \tag{63}$$

where $k = 1, 2, \ldots, M$. In terms of these variables one can rewrite Govorkov trilinear and bilinear relations (1)–(5) in an equivalent form

$$[\beta_\lambda, [\beta_\mu, \beta_\nu]] = \delta_{\lambda\mu}\beta_\nu - \delta_{\lambda\nu}\beta_\mu, \tag{64}$$

$$[\zeta_\lambda, [\zeta_\mu, \zeta_\nu]] = \delta_{\lambda\mu}\,\zeta_\nu - \delta_{\lambda\nu}\,\zeta_\mu, \tag{65}$$

$$[\zeta_\lambda, [\zeta_\mu, \beta_\nu]] = 2\delta_{\mu\nu}\beta_\lambda + \delta_{\lambda\nu}\beta_\mu + \delta_{\lambda\mu}\beta_\nu, \tag{66}$$

$$[\beta_\lambda, [\zeta_\mu, \beta_\nu]] = -2\delta_{\mu\nu}\zeta_\lambda - \delta_{\lambda\nu}\zeta_\mu - \delta_{\lambda\mu}\zeta_\nu, \tag{67}$$

$$[\beta_\mu, \beta_\nu] = [\zeta_\mu, \zeta_\nu], \quad [\zeta_\mu, \beta_\nu] = [\zeta_\nu, \beta_\mu]. \tag{68}$$

A connection with the Duffin-Kemmer-Petiau formalism

It is necessary to verify whether the relations (64) – (68) will be satisfied if we stay only within the framework of the DKP formalism. By virtue of the DKP algebra the bilinear relations (68) are fulfilled. However, the second bilinear relation in (68) within DKP theory has actually more "weak" form:

$$[\zeta_\mu, \beta_\nu] = [\zeta_\nu, \beta_\mu] = -i\omega\delta_{\mu\nu}. \tag{69}$$

It is precisely this circumstance that has negative consequence for the trilinear relations which is now under consideration.

The trilinear relation (64) is satisfied by virtue of the DKP algebra. The relation (65) also holds, since the same algebra is true for the matrices ζ_μ. On the strength of (69), we now have for (66)

$$[\zeta_\lambda, [\zeta_\mu, \beta_\nu]] = -i\delta_{\mu\nu}[\zeta_\lambda, \omega] \equiv \delta_{\mu\nu}\beta_\lambda,$$

but there should be

$$[\zeta_\lambda, [\zeta_\mu, \beta_\nu]] = 2\delta_{\mu\nu}\beta_\lambda + \delta_{\lambda\nu}\beta_\mu + \delta_{\lambda\mu}\beta_\nu.$$

Thus, there is a significant difference between the right-hand sides of these commutators. The trilinear relation (66), just as (67), is not satisfied. We can summarize these considerations with the statement that:
in spite of a close similarity between these two formalisms, the scheme of quantization based on the Duffin-Kemmer-Petiau theory does not embed into the scheme of the unitary quantization suggested by Govorkov.

Conclusion

- In this talk we have considered various aspects of a connection between the unitary quantization and parastatistics. In the analysis of the connection, the primary emphasis has been placed on the use of the Green decomposition of the creation and annihilation operators, and also para-Grassmann numbers.
- It was found that a system of the commutation relations derived by Govorkov in the framework of uniquantization is very severe, since it has been possible to associate this system only with a particular case of parastatistics, namely, with the para-Fermi statistics of order 2. However, even so, we needed to introduce a number of additional assumptions and a new operator Ω.
- In the paper by A.B. Govorkov (1979) the case of an odd number of dimensions, i.e. the unitary group $SU(2M)$, was also considered. It has been shown that the Lie algebra of the unitary group contains the Lie algebra of the symplectic group $Sp(2M)$. The quantization in accordance with the Lie algebra of the symplectic group $Sp(2M)$ corresponds to paraboson quantization. One can state a similar task of the connection between the unitary quantization scheme based on the Lie algebra of the unitary group $SU(2M)$ and para-Bose statistics.

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Thanks for attention!

The Lie algebra of the unitary group $SU(2M+1)$ has the form

$$[X_{\mu\nu}, X_{\sigma\lambda}] = \delta_{\nu\sigma} X_{\mu\lambda} - \delta_{\mu\lambda} X_{\sigma\nu}, \quad \sum_{\mu=0}^{2M} X_{\mu\mu} = 0, \tag{70}$$

where the indices $\mu, \nu, \ldots$ run values $0, 1, 2, \ldots 2M$. By introducing a new set of operators

$$F_{\mu\nu} = X_{\mu\nu} - X_{\nu\mu}, \quad F_{\mu\nu} = -F_{\nu\mu}$$

$$\widetilde{F}_{\mu\nu} = X_{\mu\nu} + X_{\nu\mu}, \quad \widetilde{F}_{\mu\nu} = +\widetilde{F}_{\nu\mu},$$

the Lie algebra (70) becomes

$$[F_{\mu\nu}, F_{\sigma\lambda}] = \delta_{\nu\sigma} F_{\mu\lambda} + \delta_{\mu\lambda} F_{\nu\sigma} - \delta_{\mu\sigma} F_{\nu\lambda} - \delta_{\nu\lambda} F_{\mu\sigma}, \tag{71}$$

$$[\widetilde{F}_{\mu\nu}, \widetilde{F}_{\sigma\lambda}] = \delta_{\nu\sigma} F_{\mu\lambda} + \delta_{\mu\lambda} F_{\nu\sigma} + \delta_{\mu\sigma} F_{\nu\lambda} + \delta_{\nu\lambda} F_{\mu\sigma}, \tag{72}$$

$$[F_{\mu\nu}, \widetilde{F}_{\sigma\lambda}] = \delta_{\nu\sigma} \widetilde{F}_{\mu\lambda} - \delta_{\mu\lambda} \widetilde{F}_{\nu\sigma} - \delta_{\mu\sigma} \widetilde{F}_{\nu\lambda} + \delta_{\nu\lambda} \widetilde{F}_{\mu\sigma}. \tag{73}$$

The condition of speciality turns into: $\sum_{\mu=0}^{2M} \widetilde{F}_{\mu\mu} = 0$. The operators $F_{\mu\nu}$ form the Lie algebra of the orthogonal group $SO(2M+1)$ and the operators $\widetilde{F}_{\mu\nu}$ complete this algebra to the algebra of the unitary group $SU(2M+1)$.

The unitary quantization procedure is based on the choice of the Lie algebra of the group $SO(2M+1)$ as the basis algebra. Further, we introduce the following quantities:

$$\begin{aligned} \beta_\mu &\equiv iF_{\mu 0}, \quad \beta_0 = iF_{00} = 0, \\ \zeta_\mu &\equiv \widetilde{F}_{\mu 0}, \quad \zeta_0 = \widetilde{F}_{00} \neq 0. \end{aligned} \tag{74}$$

In terms of the variables (74) one can rewrite the algebra (71)–(73) in an equivalent form of the trilinear relations

$$[\beta_\lambda, [\beta_\mu, \beta_\nu]] = \delta_{\lambda\mu}\beta_\nu - \delta_{\lambda\nu}\beta_\mu,$$

$$[\zeta_\lambda, [\zeta_\mu, \zeta_\nu]] = \delta_{\lambda\mu}\,\zeta_\nu - \delta_{\lambda\nu}\,\zeta_\mu,$$

$$[\zeta_\lambda, [\zeta_\mu, \beta_\nu]] = 2\delta_{\mu\nu}\beta_\lambda + \delta_{\lambda\nu}\beta_\mu + \delta_{\lambda\mu}\beta_\nu,$$

$$[\beta_\lambda, [\zeta_\mu, \beta_\nu]] = -2\delta_{\mu\nu}\zeta_\lambda - \delta_{\lambda\nu}\zeta_\mu - \delta_{\lambda\mu}\zeta_\nu,$$

and the bilinear ones

$$[\beta_\mu, \beta_\nu] = [\zeta_\mu, \zeta_\nu],$$

$$[\zeta_\mu, \beta_\nu] = [\zeta_\nu, \beta_\mu].$$

Here, the indices run values $1, 2, \ldots, 2M$.

The Lie algebra $su(2M+1)$

The generalization of the trilinear relations valid for any values of indices takes the form, correspondingly,

$$[\beta_\lambda, [\beta_\mu, \beta_\nu]] = \delta_{\lambda\mu}\beta_\nu - \delta_{\lambda\nu}\beta_\mu + \delta_{0\nu}\big(\delta_{0\lambda}\beta_\mu - \delta_{0\mu}\beta_\lambda\big) - \delta_{0\mu}\big(\delta_{0\lambda}\beta_\nu - \delta_{0\nu}\beta_\lambda\big),$$

$$[\zeta_\lambda, [\zeta_\mu, \zeta_\nu]] = \delta_{\lambda\mu}\,\zeta_\nu - \delta_{\lambda\nu}\,\zeta_\mu + \delta_{0\nu}\big(\delta_{0\lambda}\zeta_\mu - \delta_{0\mu}\zeta_\lambda - \delta_{\mu\lambda}\zeta_0\big)$$

$$- \,\delta_{0\mu}\big(\delta_{0\lambda}\zeta_\nu - \delta_{0\nu}\zeta_\lambda - \delta_{\nu\lambda}\zeta_0\big) + 2i\big(\delta_{0\nu}[\beta_\mu, \zeta_\lambda] - \delta_{0\mu}[\beta_\nu, \zeta_\lambda]\big),$$

$$[\zeta_\lambda, [\zeta_\mu, \beta_\nu]] = 2\delta_{\mu\nu}\beta_\lambda + \delta_{\lambda\nu}\beta_\mu + \delta_{\lambda\mu}\beta_\nu - \delta_{0\nu}\big(\delta_{0\lambda}\beta_\mu - \delta_{0\mu}\beta_\lambda\big)$$

$$- \,\delta_{0\mu}\big(\delta_{0\lambda}\beta_\nu - \delta_{0\nu}\beta_\lambda\big) + 2i\delta_{0\mu}[\zeta_\nu, \zeta_\lambda].$$

Here, the indices run values $0, 1, 2, \ldots, 2M$. A distinguishing feature of the last two expressions is appearing the terms, which are *bilinear* in β and ζ operators. These terms cannot be eliminated by any means. The generalization of the bilinear relations are

$$[\zeta_\mu, \zeta_\nu] = [\beta_\mu, \beta_\nu] - 2i\big(\delta_{0\nu}\beta_\mu - \delta_{0\mu}\beta_\nu\big),$$

$$[\zeta_\mu, \beta_\nu] - [\zeta_\nu, \beta_\mu] = 2i\big(\delta_{0\nu}\zeta_\mu - \delta_{0\mu}\zeta_\nu\big)$$

and in particular, for $\nu = 0$ we have (recall that $\beta_0 = 0$)

$$[\zeta_\mu, \zeta_0] = -2i\beta_\mu, \quad [\beta_\mu, \zeta_0] = 2i\zeta_\mu. \tag{75}$$

For the unitary representations of the algebra under consideration, the quantities β_μ and ζ_μ are Hermitian: V

$$\beta_\mu^\dagger = \beta_\mu, \quad \zeta_\mu^\dagger = \zeta_\mu.$$

This circumstance enables us to introduce Hermitian conjugate operators

$$\begin{aligned} a_k &= \beta_{2k-1} - i\beta_{2k}, \quad & b_k &= \zeta_{2k-1} - i\zeta_{2k}, \\ a_k^\dagger &= \beta_{2k-1} + i\beta_{2k}, \quad & b_k^\dagger &= \zeta_{2k-1} + i\zeta_{2k}, \end{aligned} \tag{76}$$

where $k = 1, 2, \ldots, M$. The algebra (1) – (5) and

$$[\hat{a}_k, \zeta_0] = 2i\hat{b}_k, \quad [\hat{b}_k, \zeta_0] = -2i\hat{a}_k,$$

where

$$\zeta_0 = \frac{i}{2(2M+1)} \sum_{k=1}^{M} \Big([a_k^\dagger, b_k] + [b_k^\dagger, a_k] \Big)$$

for the operators a_k, b_k and ζ_0 is a direct corollary of (64) – (68) and (75). In terms of the operator ζ_0, the operator $\widetilde{N}$ (28) can be also presented in the form

$$\widetilde{N} = \frac{1}{2}\,\zeta_0 + \frac{iM}{2(2M+1)}\,\lambda. \tag{77}$$

The mapping of the trilinear relations

Let us discuss a mapping of the relations, where this circumstance doesn't take place, for example, the mapping of the relations of the form

$$\{\xi_l, [b_m^\dagger, a_k]\} = 2\bigl(\Lambda - \Lambda^*\bigr)\delta_{mk}\xi_l, \quad [\xi_l, \{a_k^\dagger, b_m\}] = 0.$$

It is evident that under the mapping (33) these two relations can never go over into each other, since their right-hand sides are different. By repeating the above arguments, we obtain

$$\{\xi_l, [a_m^\dagger, b_k]\} = 2\bigl(\Lambda^* - \Lambda\bigr)\delta_{mk}\xi_l, \quad [\xi_l, \{b_k^\dagger, a_m\}] = 0,$$

i.e the structure of the trilinear relations remains unchanged. The same is true for the trilinear relations, which don't contain the variable ξ_k at all, for example,

$$[[a_k^\dagger, a_l], b_m] = -2\delta_{km}b_l \quad \Rightarrow \quad [[b_k^\dagger, b_l], a_m] = -2\delta_{km}a_l.$$

Here, the structure of the relation is completely conserved with the only replacement $a_m \to b_m$.

Thus, all trilinear commutation relations break up into two sets, one of which changes its structure under the mapping (33), and the other conserves it. Everything depends on how the para-Grassmann variable ξ_k enters into the specific trilinear relation.