# Higher order wave equation in Bhabha-Madhavarao spin 3/2 theory

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# Introduction

In this talk we would like to expand the ideas of our early work (Yu. Markov *et al.* (2015)) concerning with the case of a spin-1 massive particle within the Duffin-Kemmer-Petiau formalism to the case of a massive particle with spin 3/2. There are a long history of the spin-3/2 theory and a substantial body of publications starting with the pioneer papers by P. Dirac (1936), M. Fierz and W. Pauli (1939).

H.J. Bhabha (1945) has offered own theory for relativistic particles of any spin and, in particular, of the spin 3/2. Bhabha has studied in detail the algebraic aspects of a first order wave equation in the form:

$$(\beta_\mu \partial_\mu + mI)\Psi(x) = 0,$$

with the only assumption that the transformation properties of the wave function, and hence the spin of the particle, are determined entirely by the infinitesimal transformations $I_{\mu\nu}$ given by the following expression: $I_{\mu\nu} = [\beta_\mu, \beta_\nu]$. Equivalently, the $\beta$-matrices must satisfy

$$[[\beta_\mu, \beta_\nu], \beta_\lambda] = \beta_\mu \delta_{\nu\lambda} - \beta_\nu \delta_{\mu\lambda}$$

for all spins. However, it was found that it has been extremely difficult task to find the explicit expressions for algebras, to which the matrices $\beta_\mu$ in Bhabha's theory have to satisfy. B.S. Madhavarao (1947) was the first who has defined an explicit form of these algebras for the special cases of the 3/2 and 2 spins. Many years later, some particular commutation relations of the $\beta$-matrices were derived by H.J. Baisya (1995) for the case of spin 5/2.

# The algebra Bhabha-Madhavarao

For the spin 3/2, the algebra Bhabha-Madhavarao is of the following form:

$$
\begin{aligned}
2\big(\beta_\mu\beta_\nu\beta_\lambda\beta_\sigma + \beta_\mu\beta_\sigma\beta_\lambda\beta_\nu + \beta_\nu\beta_\lambda\beta_\sigma\beta_\mu + \beta_\sigma\beta_\lambda\beta_\nu\beta_\mu\big) \\
= 3\big(\beta_\mu\beta_\nu + \beta_\nu\beta_\mu\big)\delta_{\lambda\sigma} + 3\big(\beta_\mu\beta_\sigma + \beta_\sigma\beta_\mu\big)\delta_{\nu\lambda} \\
+ \big(\beta_\lambda\beta_\sigma + \beta_\sigma\beta_\lambda\big)\delta_{\mu\nu} + \big(\beta_\lambda\beta_\nu + \beta_\nu\beta_\lambda\big)\delta_{\mu\sigma} \\
+ \big(\beta_\nu\beta_\sigma + \beta_\sigma\beta_\nu\big)\delta_{\mu\lambda} + \big(\beta_\mu\beta_\lambda + \beta_\lambda\beta_\mu\big)\delta_{\nu\sigma} \\
- \frac{3}{2}\big(\delta_{\mu\nu}\delta_{\lambda\sigma} + \delta_{\mu\lambda}\delta_{\nu\sigma} + \delta_{\nu\lambda}\delta_{\mu\sigma}\big)I.
\end{aligned}
\tag{1}
$$

The algebra has considerably more complicated structure in comparison with the Duffin-Kemmer-Petiau algebra for the spin-1 particles, and is its immediate extension to the case of spin 3/2. The structure of this algebra perfectly coincides with the structure of the so-called para-Fermi algebra of order $p = 3$ obtained by S. Kamefuchi, Y. Takahashi (1962), and H. Scharfstein (1963).

This circumstance may be very helpful in the construction of the path integral representation for the spin-3/2 particle propagator interacting with a background gauge field that we will briefly discuss at the end of the talk.

It was shown (B.S. Madhavarao *et al.* (1946)) the algebra (1) is a direct product of the corresponding Clifford-Dirac algebra $D_\gamma$ and the algebra called $A_\xi$-algebra generated by the matrices $\xi_\mu$. In other words the matrices $\beta_\mu$ can be presented as

$$\beta_\mu = \gamma_\mu \otimes \xi_\mu \qquad \text{(no summation!)}. \qquad (2)$$

Here, $\gamma_\mu$ is usual (Euclide) $4 \times 4$ Dirac matrices. The algebraic relations for the $\xi_\mu$ matrices are given below

$$\xi_\mu^2 = \xi_\mu + \frac{3}{4}, \qquad (\xi_\mu \xi_\nu + \xi_\nu \xi_\mu) + 2\xi_\mu \xi_\nu \xi_\mu = -\frac{1}{2}\,\xi_\nu, \quad (\mu \neq \nu)$$
$$(\xi_\mu \xi_\nu \xi_\lambda - \xi_\lambda \xi_\nu \xi_\mu) = (\xi_\nu \xi_\lambda \xi_\mu - \xi_\mu \xi_\lambda \xi_\nu) = (\xi_\lambda \xi_\mu \xi_\nu - \xi_\nu \xi_\mu \xi_\lambda),$$
$$\xi_\mu(\xi_\nu \xi_\lambda \xi_\sigma - \xi_\sigma \xi_\lambda \xi_\nu) = (\xi_\nu \xi_\lambda \xi_\sigma - \xi_\sigma \xi_\lambda \xi_\nu)\xi_\mu, \quad (\mu \neq \nu \neq \lambda \neq \sigma).$$

The $A_\xi$-algebra has three irreducible representations of degree 1, 4 and 5. For $D = 4$ , the total number of independent elements is equal to 42 and the center consists of three elements

$$I_\xi, \qquad P_2 - P_1, \qquad P_4 - 2P_3, \qquad (3)$$

where $\quad P_1 = \sum \xi_\mu,\ P_2 = \sum \xi_\mu \xi_\nu,\ P_3 = \sum \xi_\mu \xi_\nu \xi_\lambda,\ P_4 = \sum \xi_\mu \xi_\nu \xi_\lambda \xi_\sigma.$

## The fourth-order wave operator

If one takes as a general guiding principle the considerations for the spin-1 case (Yu. Markov *et al.* (2015)), then the next step to the spin-3/2 case will be the following extension: as a basis we take the fourth roots of unity $(q,\, q^2,\, q^3,\, 1)$, where

$$q = i, \qquad q^2 = -1, \qquad q^3 = -i. \tag{5}$$

In the same way as the spin 1/2 and 1 cases, the starting point of all further considerations will be the following expression for the fourth-order massive wave operator:

$$\big[(\beta{\cdot}\partial)+qmI\big]\big[(\beta{\cdot}\partial)+q^2mI\big]\big[(\beta{\cdot}\partial)+q^3mI\big]\big[(\beta{\cdot}\partial)+mI\big] = (\beta{\cdot}\partial)^4 - m^4 I.$$

Here, we have used one of the properties of the forth roots of unity, namely,

$$1 + q^2 = 0, \qquad q + q^3 = 0. \tag{6}$$

In view of the algebra (1) the right-hand side can be presented as:

$$(\beta\cdot\partial)^4 - m^4 I \equiv \frac{5}{2}\,(\beta\cdot\partial)^2\Box - \frac{9}{16}\,\Box^2 I - m^4 I. \tag{7}$$

It is precisely this expression that we accept as the definition of the fourth order wave operator for the spin-3/2 particle.

We can state a question of defining a matrix $A$ such that

$$\left[A(\beta \cdot \partial + mI)\right]^4 = \frac{1}{m^2}\left\{\frac{5}{2}\left(\beta \cdot \partial\right)^2 \Box - \frac{9}{16}\,\Box^2 I\right\} - m^2 I. \tag{8}$$

The relation solves the problem of calculating the fourth root of the fourth order wave operator. By equating the coefficients of partial derivatives, we obtain a system of algebraic equations for the unknown matrix $A$:

$$A^4 = -\frac{1}{m^2}\,I, \tag{9}$$

$$A\beta_\mu A^3 + A^2\beta_\mu A^2 + A^3\beta_\mu A = \frac{1}{m^2}\,\beta_\mu, \tag{10}$$

$$A\beta_\mu A\beta_\nu A^2 + A\beta_\mu A^2\beta_\nu A + A^2\beta_\mu A\beta_\nu A + (\mu \rightleftarrows \nu) = -\frac{1}{m^2}\{\beta_\mu, \beta_\nu\} \tag{11}$$

and two further equations of the third and fourth degrees of nonlinearity in the $\beta$-matrices. We will show below that Eqs. (9) – (10) uniquely determine the matrix $A$. The third equation (11) and two remaining equations must be identically satisfied. If this does not hold, we come to contradiction.

## The matrix $\Omega$

Let us introduce a matrix $\Omega$ satisfying the following *characteristic equation*:

$$\Omega^4 = \frac{5}{2}\,\Omega^2 - \frac{9}{16}\,I. \qquad (12)$$

An explicit form of the matrix $\Omega$ will be defined just below. We seek the matrix $A$ in the form of the most general expansion in powers of $\Omega$:

$$A = \alpha I + \beta\Omega + \gamma\Omega^2 + \delta\Omega^3, \qquad (13)$$

where $\alpha$, $\beta$, $\gamma$, $\delta$ are complex scalar constants obeyed the nonlinear system:

$$\frac{8}{9}\,\alpha^2 - \frac{45}{32}\,\delta^2 - \frac{9}{8}\,\beta\delta = a, \qquad \frac{1}{9}\,\alpha\beta + \frac{1}{36}\,\alpha\delta = a,$$

$$\beta^2 - \frac{32}{81}\,\alpha^2 + \frac{91}{16}\,\delta^2 + 5\beta\delta = -4a, \quad \gamma = -\frac{4}{9}\,\alpha, \quad a \equiv \pm\, i\,\frac{1}{8m}.$$

One of the possible solutions has the following form:

$$\beta^{(\pm)} = \left[\frac{2}{3^3} + (\pm i)\left(2 - \frac{2}{3^3}\right)\right]\alpha, \quad \delta^{(\pm)} = \left[-\frac{8}{3^3} + (\pm i)\left(-\frac{8}{3^2} + \frac{8}{3^3}\right)\right]\alpha,$$

where the parameter $\alpha$ are fixed by the relation: $\alpha^4 = \dfrac{1}{4}\left(\dfrac{9}{8}\right)^4 \dfrac{1}{m^2}\,.$

# An explicit form of the $\Omega$ matrix

We seek for the matrix $\Omega$ in the form of the decomposition

$$\Omega = \gamma_5 \otimes \omega, \tag{14}$$

where $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ and $\omega$ is the unknown matrix. In choice of the presentation (14) the characteristic equation (12) turns into the equation for $\omega$:

$$\omega^4 - \frac{5}{2}\omega^2 + \frac{9}{16} I_\xi = 0. \tag{15}$$

Here, we have taken into account that $I = I_\gamma \otimes I_\xi$, where $I_\xi$ is the unity matrix of proper dimension of the $A_\xi$-algebra. We will search for the matrix $\omega$ as a second-order polynomial in the central element $\theta$ of the algebra $A_\xi$:

$$\omega = \mu\theta^2 + \nu\theta + \lambda I_\xi. \tag{16}$$

Here $\mu$, $\nu$ and $\lambda$ are the unknown parameters and a new matrix $\theta$ by definition equals the second central element of the algebra $A_\xi$, Eq. (3):

$$\theta \equiv P_2 - P_1. \tag{17}$$

The minimal equation to which $\theta$ satisfies has the form:

$$\theta^3 - 2\theta^2 - 15\theta = 0. \tag{18}$$

## An explicit form of the $\Omega$ matrix

By virtue of the fact that the matrix $\omega$ is made up of the elements of the center of $A_\xi$-algebra, the following relation:

$$[\xi_\mu, \omega] = 0$$

is true. For matrices $\beta_\mu$ given in the form of direct product: $\beta_\mu = \gamma_\mu \otimes \xi_\mu$, the following commutative rules

$$\{\beta_\mu, \Omega\} = 0, \qquad [\beta_\mu, \Omega^2] = 0, \qquad \{\beta_\mu, \Omega^3\} = 0 \tag{19}$$

will be a consequence of (14) and the commutativity of $\xi_\mu$ and $\omega$.

The parameters $\mu$, $\nu$ and $\lambda$ are subject to the nonlinear algebraic system:

$$19\mu^2 + \nu^2 + 4\mu\nu + 2\mu\lambda = m, \quad 30\mu^2 + 30\mu\nu + 2\nu\lambda = n, \quad \lambda^2 = l, \tag{20}$$

where as $(m, n, l)$ we can only take six certain sets. One of them is

$$m = -1/12, \qquad n = 5/12, \qquad l = 9/4. \tag{21}$$

In turn, to each set of numbers $(m, n, l)$ will correspond eight solutions of the system (20). For example, for the triple of numbers (21), one of them is

$$\lambda = 3/2, \qquad \nu = 5/12, \qquad \mu = -1/12.$$

Thus, there exists a finite number of variants of choosing the matrix $\Omega$.

## The $\eta_\mu$ matrices

We have completely solved two first matrix equations (9) and (10) and defined an explicit form of the matrix $A$. It is easy to show that three remaining equations lead to a contradiction. The "naïve" representation of the fourth root as was defined on the left-hand side of expression (8) is unsuitable. Here, we need to develop more subtle approach to solving the problem in hand.

Instead of the original matrices $\beta_\mu$ we introduce by definition the following set of matrices $\eta_\mu^{(\pm)}(z)$ depending on a complex (deformation) parameter $z$:

$$\eta_\mu^{(\pm)}(z) \equiv \left(1 - \frac{1}{8}\,z\right)\beta_\mu \pm iz\frac{9}{4}\,\Omega\beta_\mu + z\frac{1}{2}\,\Omega^2\beta_\mu \mp iz\Omega^3\beta_\mu. \qquad (22)$$

Taking successively as the deformation parameter $z$ at first the primitive root $q$ and then $q^3$ we can obtain the following relations between $A$ and $\eta_\mu^{(\pm)}$:

$$A\eta_\mu^{(\pm)}(q) = q^3\eta_\mu^{(\pm)}(q)\,A - 2\left(\Pi_{1/2}\beta_\mu\right)A, \qquad (23)$$

$$A\eta_\mu^{(\pm)}(q^3) = q\,\eta_\mu^{(\pm)}(q^3)\,A - 2\left(\Pi_{1/2}\beta_\mu\right)A, \qquad (24)$$

where we have introduced the notation: $\Pi_{1/2} \equiv \dfrac{1}{2}\left(\Omega^2 - \dfrac{1}{4}\,I\right)$.

## The projectors $\mathcal{P}_{1/2}$ and $\mathcal{P}^{(\pm)}_{3/2}(q)$

We can define a set of three matrices

$$\Pi_{1/2} = \frac{1}{2}\left(\Omega^2 - \frac{1}{4}\,I\right), \qquad \Pi^{(\pm)}_{3/2} = \pm\left(\Omega^3 - \frac{9}{4}\,\Omega\right)$$

possessing the following properties

$$\begin{aligned} &\left(\Pi_{1/2}\right)^2 = \Pi_{1/2}, && \left(\Pi^{(\pm)}_{3/2}\right)^2 = I - \Pi_{1/2}, \\ &\left(\Pi^{(\pm)}_{3/2}\right)^3 = \Pi^{(\pm)}_{3/2}, && \Pi_{1/2}\Pi^{(\pm)}_{3/2} = \Pi^{(\pm)}_{3/2}\Pi_{1/2} = 0. \end{aligned} \tag{25}$$

This set of matrices does not generate a system of the projectors. Nevertheless, one can introduce a new set of three matrices possessing all the properties of projectors. Let us introduce the notation

$$\mathcal{P}^{(\pm)}_{3/2}(q) \equiv \frac{1}{2}\left[\left(\Pi^{(\pm)}_{3/2}\right)^2 - iq\,\Pi^{(\pm)}_{3/2}\right]. \tag{26}$$

If we put $\mathcal{P}_{1/2} \equiv \Pi_{1/2}$ then a set of the matrices $(\mathcal{P}_{1/2}, \mathcal{P}^{(\pm)}_{3/2}(q))$ satisfies the standard relations of the usual algebra of projectors:

$$\begin{aligned} &\left(\mathcal{P}_{1/2}\right)^2 = \mathcal{P}_{1/2}, && \left(\mathcal{P}^{(\pm)}_{3/2}(q)\right)^2 = \mathcal{P}^{(\pm)}_{3/2}(q), \\ &\mathcal{P}^{(\pm)}_{3/2}(q)\mathcal{P}^{(\mp)}_{3/2}(q) = 0, && \mathcal{P}_{1/2}\mathcal{P}^{(\pm)}_{3/2}(q) = \mathcal{P}^{(\pm)}_{3/2}(q)\mathcal{P}_{1/2} = 0 \end{aligned} \tag{27}$$

and the completeness relation: $\mathcal{P}_{1/2} + \mathcal{P}^{(\pm)}_{3/2}(q) + \mathcal{P}^{(\mp)}_{3/2}(q) = I$.

Further, we introduce by definition the following projected $\beta$-matrices:

$$\eta_\mu^{(1/2)} \equiv \mathcal{P}_{1/2}\beta_\mu, \qquad \eta_\mu^{(\pm 3/2)}(q) \equiv \mathcal{P}_{3/2}^{(\pm)}(q)\beta_\mu. \tag{28}$$

Multiplying Eqs. (23), (24) by the projector $\mathcal{P}_{3/2}^{(\pm)}(q)$ on the left, we obtain

$$A\eta_\mu^{(\pm 3/2)}(q) = q^3\eta_\mu^{(\pm 3/2)}(q)A, \qquad A\eta_\mu^{(\pm 3/2)}(q^3) = q\eta_\mu^{(\pm 3/2)}(q^3)A. \tag{29}$$

We rewrite the matrix $\eta_\mu^{(\pm)}(z)$ depending on an arbitrary complex number $z$, Eq. (22), in terms of the projected matrices $\eta_\mu^{(1/2)}$ and $\eta_\mu^{(\pm 3/2)}(q)$:

$$\eta_\mu^{(\pm)}(z) = (z-q^2)\eta_\mu^{(1/2)} + (1-zq)\eta_\mu^{(\pm 3/2)}(q) + (1+zq)\eta_\mu^{(\mp 3/2)}(q).$$

As the spin structure $S_{\mu\nu}(q)$ we accept the following expression:

$$S_{\mu\nu}(q) \equiv \lim_{z\to q}\frac{1}{\epsilon(z)}\left[\eta_\mu^{(\pm 3/2)}(z), \eta_\nu^{(\pm 3/2)}(z)\right] \tag{30}$$

$$= \left[\eta_\mu^{(\pm 3/2)}(q), \eta_\nu^{(\mp 3/2)}(q)\right] + \left[\eta_\mu^{(\mp 3/2)}(q), \eta_\nu^{(\pm 3/2)}(q)\right].$$

Here, we have introduced by definition the following important function:

$$\epsilon(z) = (z-q)(z-q^3). \tag{31}$$

## The $\eta_\mu$ matrices: commutation relations

Let us consider the double commutation relation with the $\eta_\mu^{(\pm 3/2)}(q)$ matrices. By using (30), we have

$$\lim_{z\to q} \frac{1}{\epsilon(z)} \left[ \left[ \eta_\mu^{(\pm 3/2)}(z), \eta_\nu^{(\pm 3/2)}(z) \right], \eta_\lambda^{(\pm 3/2)}(z) \right] = \eta_\mu^{(\pm 3/2)}(q)\delta_{\nu\lambda} - \eta_\nu^{(\pm 3/2)}(q)\delta_{\mu\lambda}.$$

This is an analog of the trilinear relation for the original matrices $\beta_\mu$:

$$[[\beta_\mu, \beta_\nu], \beta_\lambda] = \beta_\mu \delta_{\nu\lambda} - \beta_\nu \delta_{\mu\lambda}.$$

This relation will assure us the relativistic covariance of the following wave equation (in the limit $z \to q$):

$$A\left[ \frac{1}{\epsilon^{1/2}(z)} \eta_\mu^{(\pm 3/2)}(z)\partial_\mu + \left( \mathcal{P}_{3/2}^{(\pm)}(q) + \mathcal{P}_{3/2}^{(\mp)}(q) \right) m \right] \psi(x; z) = 0. \qquad (32)$$

An analysis of this equation will be considered just below. In the notation of the wave function $\psi$ we have explicitly separated out the dependence on the deformation parameter $z$, and in the mass term, instead of the unity matrix $I$, we have entered a sum of projectors which single out in $\psi$ only the part connected with the spin-$3/2$ sector.

## Decomposition of the projectors $\mathcal{P}^{(\pm)}_{3/2}(q)$

We can rewrite the projectors $\mathcal{P}^{(\pm)}_{3/2}(q)$ in the following equivalent form:

$$\begin{aligned} \mathcal{P}^{(+)}_{3/2}(q) &= P_R \otimes \pi^{(+)}_{3/2}(q) + P_L \otimes \pi^{(-)}_{3/2}(q), \\ \mathcal{P}^{(-)}_{3/2}(q) &= P_R \otimes \pi^{(-)}_{3/2}(q) + P_L \otimes \pi^{(+)}_{3/2}(q), \end{aligned} \qquad (33)$$

where

$$\pi^{(\pm)}_{3/2}(q) \equiv \frac{1}{2}\left(-\pi_{1/2} + I_\xi \mp iq\pi_{3/2}\right),$$

and

$$\begin{cases} P_L = \dfrac{1}{2}\left(I_\gamma - \gamma_5\right), \\ P_R = \dfrac{1}{2}\left(I_\gamma + \gamma_5\right). \end{cases} \qquad \begin{cases} \pi_{1/2} \equiv \dfrac{1}{2}\left(\omega^2 - \dfrac{1}{4}\, I_\xi\right), \\ \pi^{(\pm)}_{3/2} = \omega^3 - \dfrac{9}{4}\,\omega. \end{cases} \qquad (34)$$

Here, $P_L$ and $P_R$ are the chiral projector operators. Expressions (33) possess a remarkable feature: by virtue of the properties

$$\pi^{(\pm)}_{3/2}(q)\,\pi^{(\mp)}_{3/2}(q) = 0, \qquad P_L P_R = P_R P_L = 0,$$

the four terms on right-hand sides of (33) are orthogonal among themselves!

# Decomposition of the matrices $\eta_\mu^{(\pm 3/2)}(q)$

As a consequence, the basic matrices $\eta_\mu^{(\pm 3/2)}(q)$ can be presented in a more descriptive form:

$$\eta_\mu^{(+3/2)}(q) = \big(P_R \gamma_\mu\big) \otimes \big(\pi_{3/2}^{(+)}(q)\xi_\mu\big) + \big(P_L \gamma_\mu\big) \otimes \big(\pi_{3/2}^{(-)}(q)\xi_\mu\big),$$

$$\eta_\mu^{(-3/2)}(q) = \big(P_R \gamma_\mu\big) \otimes \big(\pi_{3/2}^{(-)}(q)\xi_\mu\big) + \big(P_L \gamma_\mu\big) \otimes \big(\pi_{3/2}^{(+)}(q)\xi_\mu\big).$$

The existence of two projectors $\mathcal{P}_{3/2}^{(+)}(q)$ and $\mathcal{P}_{3/2}^{(-)}(q)$ for the same spin-3/2 sector indicates that in the system under consideration there exists another additional internal degree of freedom (and a quantum number associated with it). This degree of freedom arises by virtue of introducing an additional algebraic object, namely a system of the roots of unity containing two primitive ones $q$ and $q^3$ in the spin-3/2 case.

## The general structure of a solution of the first-order differential equation

We analyze now the general structure of a solution of the wave equation (32) which we present as follows:

$$\hat{\mathcal{L}}^{(3/2)}(z,\partial)\psi(x;z) = 0. \tag{35}$$

Here, we have introduced a short-hand notation for the differential operator

$$\hat{\mathcal{L}}^{(3/2)}(z;\partial) \equiv A\Bigg[\frac{1}{\epsilon^{1/2}(z)}\,\eta_\mu^{(\pm 3/2)}(z)\,\partial_\mu + \Big(\mathcal{P}^{(\pm)}_{3/2}(q) + \mathcal{P}^{(\mp)}_{3/2}(q)\Big)\,m\Bigg]. \tag{36}$$

The solution of Eq. (35) can be presented in the following form:

$$\psi(x;z) = \big[\hat{\mathcal{L}}^{(3/2)}(z;\partial)\big]^3\varphi(x;z), \tag{37}$$

where in turn the function $\varphi(x;z)$ is a solution of the fourth-order wave equation

$$\big[\hat{\mathcal{L}}^{(3/2)}(z;\partial)\big]^4\varphi(x;z) = 0. \tag{38}$$

The solution $\varphi(x;z)$ is regular at $z = q$ and it can be presented in the form of a formal series expansion in positive powers of $\delta^{1/2}$ $\big(\equiv (z-q)^{1/2}\big)$ :

$$\varphi(x;z) = \varphi_0(x) + \delta^{1/2}\varphi_{\frac{1}{2}}(x) + \delta\varphi_1(x) + \delta^{3/2}\varphi_{\frac{3}{2}}(x)\; + \;\ldots\,. \tag{39}$$

Substituting the expansion (39) into the relation (37) and collecting terms of the same power in $\delta^{1/2}$, we obtain

$$\psi(x;z) = \frac{1}{\delta^{1/2}}\,\psi_{-\frac{1}{2}}(x) + \psi_0(x) + \delta^{1/2}\psi_{\frac{1}{2}}(x) + \,\ldots\,, \tag{40}$$

where

$$\psi_{-\frac{1}{2}}(x) = 2\,\frac{1}{\varrho^{1/2}}\,A^3\biggl\{\frac{2}{\varrho}\,q^2\bigl[\eta_\mu^{(\pm 3/2)}(q)\,\eta_\nu^{(\mp 3/2)}(q)\,\eta_\lambda^{(\pm 3/2)}(q)\bigr]\partial_\mu\partial_\nu\partial_\lambda \tag{41}$$

$$+\,q\,m^2\eta_\mu^{(\pm 3/2)}(q)\,\partial_\mu\biggr\}\varphi_0(x), \qquad \varrho \equiv q - q^3$$

and so on. The wave function $\psi(x;z)$ is not a regular function of the parameter $z$. By virtue of the completeness condition we may present function $\varphi_0(x)$ as follows

$$\varphi_0(x) = \varphi_0^{(1/2)}(x) + \varphi_0^{(\pm 3/2)}(x;q) + \varphi_0^{(\mp 3/2)}(x;q), \tag{42}$$

where the projected wave functions are

$$\varphi_0^{(1/2)}(x) \equiv \mathcal{P}_{1/2}\,\varphi_0(x), \qquad \varphi_0^{(\pm 3/2)}(x;q) \equiv \mathcal{P}_{3/2}^{(\pm)}(q)\varphi_0(x).$$

On the right-hand side of the expression (41) in the decomposition (42) only one of the projected parts, namely $\varphi_0^{(\mp 3/2)}(x;q)$, survives. The singular contribution in the expansion (40) can be dropped out, if we simply set: $\varphi_0^{(\mp 3/2)}(x;q) \equiv 0$.

In this case, as $z \to q$, the first nonzero contribution in $\psi(x;q)$ has the form

$$\psi_0(x) = -mA^3\left\{\frac{2}{\varrho}(2+q)\left[\eta_\mu^{(\pm 3/2)}(q)\eta_\nu^{(\mp 3/2)}(q)\right]\partial_\mu\partial_\nu - m^2 I\right\}\varphi_0^{(\pm 3/2)}(x;q) +$$

$$\frac{2A^3}{\varrho^{1/2}}\left\{\frac{2}{\varrho}q^2\left[\eta_\mu^{(\pm 3/2)}(q)\eta_\nu^{(\mp 3/2)}(q)\eta_\lambda^{(\pm 3/2)}(q)\right]\partial_\mu\partial_\nu\partial_\lambda + m^2\eta_\mu^{(\pm 3/2)}(q)\partial_\mu\right\}\varphi_{\frac{1}{2}}^{(\mp 3/2)}(x;q).$$

The differential equations to which the functions $\varphi_0(x)$, $\varphi_{\frac{1}{2}}(x)$, ... must satisfy, are defined by the appropriate expansion of the operator $[\hat{\mathcal{L}}^{(3/2)}(z;\partial)]^4$. We have the following equation to leading order in $\delta^{1/2}$:

$$\delta^0 : \left\{-\frac{1}{m^2}\frac{1}{\varrho^2}\left\{\frac{5}{2}\left(\left[\eta_\mu^{(\pm 3/2)}(q)\eta_\nu^{(\mp 3/2)}(q)\right]\partial_\mu\partial_\nu\right)\Box - \frac{9}{16}\Box^2\mathcal{P}_{3/2}^{(\pm)}(q)\right\}\right.$$
$$\left. + \frac{1}{\rho}(1+q)\left[\eta_\mu^{(\pm 3/2)}(q)\eta_\nu^{(\mp 3/2)}(q)\right]\partial_\mu\partial_\nu - \frac{1}{4}m^2 I\right\}\varphi_0^{(\pm 3/2)}(x;q) \quad (43)$$
$$= -2\frac{1}{m}\frac{1}{\varrho^{3/2}}(1+q)\left[\eta_\mu^{(\pm 3/2)}(q)\eta_\nu^{(\mp 3/2)}(q)\eta_\lambda^{(\pm 3/2)}(q)\right]\partial_\mu\partial_\nu\partial_\lambda\,\varphi_{\frac{1}{2}}^{(\mp 3/2)}(x;q).$$

The equation for the function $\varphi_0^{(\pm 3/2)}(x;q)$ is nonclosed. We can show that the only restriction: $\varphi_0^{(\mp 3/2)}(x;q) \equiv 0$, finally leads to closed completely self-consistent calculation scheme of the wave function $\psi(x;z)$ regular in the limit $z \to q$.

Let us consider the question of a modification of the fourth-order wave operator (43) in the presence of an external electromagnetic field. We introduce the interaction via the minimal substitution:

$$\partial_\mu \to D_\mu \equiv \partial_\mu + ieA_\mu(x).$$

With an external gauge field in the system the left-hand side of (43) takes the form

$$\left[\eta^{(\pm 3/2)}_{\mu_1}(q)\eta^{(\mp 3/2)}_{\mu_2}(q)\eta^{(\pm 3/2)}_{\mu_3}(q)\eta^{(\mp 3/2)}_{\mu_4}(q)\right] D_{\mu_1}D_{\mu_2}D_{\mu_3}D_{\mu_4}. \tag{44}$$

For analysis of the expression (44) we make use of the following identity for product of four covariant derivatives

$$\begin{aligned} D_{\mu_1}D_{\mu_2}D_{\mu_3}D_{\mu_4} = \frac{1}{4!}\Big(&\{D_{\mu_1},D_{\mu_2},D_{\mu_3},D_{\mu_4}\} + 12ieD_{\mu_1}D_{\mu_2}F_{\mu_3\mu_4} + 4ieD_{\mu_1}D_{\mu_3}F_{\mu_2\mu_4} \\ &+ 2ie\big(D_{\mu_2}D_{\mu_3}F_{\mu_1\mu_4} + D_{\mu_2}D_{\mu_4}F_{\mu_1\mu_3} + D_{\mu_3}D_{\mu_4}F_{\mu_1\mu_2}\big) \\ &+ 6ie\big(F_{\mu_1\mu_2}D_{\mu_3}D_{\mu_4} + F_{\mu_1\mu_3}D_{\mu_2}D_{\mu_4} + F_{\mu_1\mu_4}D_{\mu_2}D_{\mu_3}\big) \\ &+ 8ieD_{\mu_1}F_{\mu_2\mu_3}D_{\mu_4} + 2ie\sum\nolimits_{(\mathcal{P})} D_{\mu_2}F_{\hat{\mu}_1\mu_3}D_{\mu_4} \\ &+ 4e^2\big(F_{\mu_1\mu_2}F_{\mu_3\mu_4} + F_{\mu_1\mu_3}F_{\mu_2\mu_4} + F_{\mu_1\mu_4}F_{\mu_2\mu_3}\big)\Big), \end{aligned} \tag{45}$$

where $\{D_{\mu_1},\ldots,D_{\mu_4}\}$ denotes the completely symmetrized product of four $D$-operators and $F_{\mu_1\mu_2}(x)$ is the strength tensor.

The contribution in (44) due to the symmetrized part can be presented as

$$\left[\eta_{\mu_1}^{(\pm 3/2)}(q)\eta_{\mu_2}^{(\mp 3/2)}(q)\eta_{\mu_3}^{(\pm 3/2)}(q)\eta_{\mu_4}^{(\mp 3/2)}(q)\right]\{D_{\mu_1},D_{\mu_2},D_{\mu_3},D_{\mu_4}\} \tag{46}$$

$$=\left(\frac{5}{2}\left[\eta_{\mu_1}^{(\pm 3/2)}(q)\eta_{\mu_2}^{(\mp 3/2)}(q)\right]\delta_{\mu_3\mu_4}+\frac{9}{16}\,\delta_{\mu_1\mu_2}\delta_{\mu_3\mu_4}\right)\{D_{\mu_1},D_{\mu_2},D_{\mu_3},D_{\mu_4}\}.$$

The contraction of the term $\delta_{\mu_1\mu_2}\delta_{\mu_3\mu_4}$ with $\{D_{\mu_1},D_{\mu_2},D_{\mu_3},D_{\mu_4}\}$ gives us:

$$\{D_{\mu_1},D_{\mu_2},D_{\mu_3},D_{\mu_4}\}\delta_{\mu_1\mu_2}\delta_{\mu_3\mu_4}=24D^4-8ieD_\mu F_{\mu\nu}D_\nu-4ieD_\mu D_\nu F_{\mu\nu}. \tag{47}$$

Further we must proceed to the consideration of the contraction with the term containing the matrices $\eta_{\mu}^{(\pm 3/2)}(q)$, namely,

$$\left[\eta_{\mu_1}^{(\pm 3/2)}(q)\eta_{\mu_2}^{(\mp 3/2)}(q)\right]\delta_{\mu_3\mu_4}\{D_{\mu_1},D_{\mu_2},D_{\mu_3},D_{\mu_4}\}. \tag{48}$$

The expressions here are already more cumbersome and tangled. For these reason, we do not present further details. The obtained expressions (47) and (48) will be completely define the contraction (46) with the totally symmetrized product.

We need to consider a similar contraction with the remaining terms on the r.h.s. of the identity (45). By virtue of awkwardness of the expressions we do not give their final form. We note only that here the contractions of strength tensor $F_{\mu\nu}(x)$ with the spin structure $S_{\mu\nu}(q)$ will take place, as it was defined by the expression (30).

We have early discussed (Yu. Markov *et al.* (2015)) a fundamental difficulty connected with the construction of the path integral representation for the spin-1 massive particle propagator interacting with a background gauge field within the Duffin-Kemmer-Petiau theory. This difficulty is closely related to *noncommutativity* of the differential operator in the presence of an electromagnetic field

$$L(D) = \beta_\mu D_\mu + mI, \tag{49}$$

and the proper divisor

$$d(D) = \frac{1}{m}(D^2 - m^2)I + \beta_\mu D_\mu - \frac{1}{m}\beta_\mu\beta_\nu D_\mu D_\nu.$$

In these expressions the matrices $\beta_\mu$ satisfy the trilinear relation:

$$\beta_\mu\beta_\nu\beta_\lambda + \beta_\lambda\beta_\nu\beta_\mu = \delta_{\mu\nu}\beta_\lambda + \delta_{\lambda\nu}\beta_\mu.$$

A similar situation will take place and for the spin-3/2 case, where now in the operator (49) the matrices $\beta_\mu$ satisfy Bhabha-Madhavarao (1) and the proper divisor should be taken in the form:

$$d(D) = -\frac{16}{9}\left[m^3 - m^2(\beta\cdot D) + m(\beta\cdot D)^2 - \frac{5}{2}\,mD^2 I - (\beta\cdot D)^3 + \frac{5}{2}\,(\beta\cdot D)D^2\right].$$

To circumvent the difficulty connected with noncommutativity in constructing the path integral representation for the spin-3/2 massive particle propagator in the presence of an external gauge field, we can proceed in complete analogy to the spin-1 case. In the case of the Bhabha-Madhavarao theory as a basic element of the construction, we take the fourth root of the fourth order wave operator

$$\hat{\mathcal{L}}(z, D) = A\left[\frac{1}{\epsilon^{1/2}(z)}\,\eta_{\mu}^{(\pm 3/2)}(z)\,D_{\mu} + \Big(\mathcal{P}_{3/2}^{(\pm)}(q) + \mathcal{P}_{3/2}^{(\mp)}(q)\Big)\,m\right].$$

Let us assume that $\hat{\mathcal{L}}(z, D)$ is a para-Fermi operator (parastatistics of order 3). The Fock-Schwinger proper-time representation for the inverse operator $\hat{\mathcal{L}}^{-1}$ is

$$\frac{1}{\hat{\mathcal{L}}} \equiv \frac{\hat{\mathcal{L}}^3}{\hat{\mathcal{L}}^4} = i\int_0^{\infty} d\tau \int \frac{d^3\chi}{\tau^3}\ \mathrm{e}^{-i\tau\left(\hat{H}(z)-i\epsilon\right)+\frac{1}{2}\left(\tau[\chi,\hat{\mathcal{L}}]+\frac{1}{4}\tau^2[\chi,\hat{\mathcal{L}}]^2-\frac{5}{12}\tau^3\,[\chi,\hat{\mathcal{L}}]^3\right)}.$$

Here $\hat{H}(z) \equiv \hat{\mathcal{L}}^4(z, D)$ and $\chi$ is para-Grassmann variable of order $p = 3$ (i.e. $\chi^4 = 0$) with the rules of an integration

$$\int d^3\chi = \int d^3\chi\,[\chi, \hat{\mathcal{L}}] = \int d^3\chi\,[\chi, \hat{\mathcal{L}}]^2 = 0, \quad \int d^3\chi\,[\chi, \hat{\mathcal{L}}]^3 = -8\hat{\mathcal{L}}^3.$$

As a proper para-supertime here it is necessary to take a tetrad $(\tau, \chi, \chi^2, \chi^3)$. This expression can be taken as the starting one for the construction of the desired path integral representation with the use of an appropriate system of coherent states. On constructing it is necessary to passage to the limit $z \to q$.

## Conclusion

- In the present work we have set up the formalism needed to construct a fourth root of the fourth order wave operator within the framework of Bhabha-Madhavarao spin-3/2 theory. The fundamental point here is the introduction of the so-called *deformed* commutator

$$[A, \beta_\mu]_z \equiv A\beta_\mu - z\beta_\mu A. \tag{50}$$

- By means of (50), instead of the original matrices $\beta_\mu$, a new set of the matrices $\eta_\mu$ depending on $z$ was defined.
- It is shown that in terms of the $\eta_\mu$ matrices we have succeeded in reducing a procedure of the construction of fourth root of the fourth order wave operator to a few simple algebraic transformations and to some operation of the passage to the limit $z \to q$.
- A set of the matrices $\mathcal{P}_{1/2}$ and $\mathcal{P}^{(\pm)}_{3/2}(q)$ possessing the properties of projectors is introduced. These operators project the matrices $\eta_\mu$ onto the spins 1/2- and 3/2-sectors in the theory under consideration.
- The application to the problem of the construction of the path integral representation in parasuperspace for the propagator of a massive spin-$\frac{3}{2}$ particle in a background gauge field within the Bhabha-Madhavarao approach was discussed.

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Thanks for attention!

## The $A_\xi$ matrix algebra

The $A_\xi$-algebra has three irreducible representations of degree 1, 4 and 5, respectively. For completeness below we give an explicit form of the matrix representation of degree 4 of $A_\xi$ in which the matrix $\xi_4$ is diagonal

$$\xi_1 = \begin{pmatrix} c & \frac{1}{2} & 0 & c \\ \frac{1}{2} & -s & 0 & s \\ 0 & 0 & -\frac{1}{2} & 0 \\ c & s & 0 & 0 \end{pmatrix}, \qquad \xi_2 = \begin{pmatrix} -s & 0 & \frac{1}{2} & s \\ 0 & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & c & c \\ s & 0 & c & 0 \end{pmatrix},$$

$$\xi_3 = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ 0 & c & \frac{1}{2} & c \\ 0 & \frac{1}{2} & -s & s \\ 0 & c & s & 0 \end{pmatrix}, \qquad \xi_4 = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{3}{2} \end{pmatrix},$$

where

$$s \equiv \sin\frac{\pi}{10} = \frac{\sqrt{5}-1}{4}, \qquad c \equiv \cos\frac{2\pi}{10} = \frac{\sqrt{5}+1}{4}.$$

An explicit form of the representation of degree 5 of $A_\xi$, in which the matrix $\xi_4$ is diagonal, was also derived (B.S. Madhavarao *et al.* (1946)) . Besides, the scheme of obtaining nondiagonal representations was presented, and spurs of elements of the basis of the $A_\xi$-algebra in the three irreducible representations was calculated.