Lorentz Covariance in Higher-Spin Equations

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Higher-Spin Theory

Higher-Spin (HS) theory

- is an interacting theory of massless fields of all spins (including gravity)
- possesses an infinite-dimensional HS gauge symmetry (String Theory as broken phase?)
- lives on Anti-de Sitter background (has no sensible flat limit with HS symmetry unbroken)
- is dual to different boundary vectorial models (Klebanov–Polyakov conjecture)
- is formulated in the form of Vasiliev equations (action is unknown $\Rightarrow$ direct check of AdS/CFT is unavailable)

HS fields and unfolded equations

- Massless field of spin-s is described by rank-s symmetric double-traceless tensor $\phi_{\underline{m}_1\dots\underline{m}_s}=\phi_{\underline{m}(s)}$ [Fronsdal, *Phys.Rev.* D18 (1978) 3624].
- Contracting with $(e_a)^{\underline{m}}(\sigma^a)^{\dot{\alpha}\beta}$: $\phi_{\underline{m}(s)}\longrightarrow\omega^{\alpha(s-1),\dot{\beta}(s-1)}=dx^{\underline{m}}\phi_{\underline{m}}{}^{\alpha(s-1),\dot{\beta}(s-1)}$
- In unfolded approach spin-s field is described by
  - 1-forms $\omega^{\alpha(n),\dot{\beta}(m)}$, $\quad n+m=2s-2$ (gauge d.o.f.: spin-s field and first $(s-1)$ on-shell derivatives)
  - 0-forms $C^{\alpha(n),\dot{\beta}(m)}$, $\quad |n-m|=2s$ (gauge-invariant d.o.f.: HS curvatures like Maxwell tensor, Weyl tensor and infinite towers of their derivatives)
- Auxiliary $sp(4)$-spinors $Y^A=\left(y^\alpha,\bar{y}^{\dot{\alpha}}\right)$ pack all spins into master-fields

$$\omega(Y|x)=\sum_{n,m}\frac{1}{n!m!}\omega_{\alpha(n),\dot{\beta}(m)}\left(y^\alpha\right)^n\left(\bar{y}^{\dot{\beta}}\right)^m,\quad C(Y|x)=\sum_{n,m}\frac{1}{n!m!}C_{\alpha(n),\dot{\beta}(m)}\left(y^\alpha\right)^n\left(\bar{y}^{\dot{\beta}}\right)^m$$

- Then nonlinear HS equations must be of the form

$$\mathrm{d}\omega+\omega*\omega=\Upsilon(\omega,\omega,C)+\Upsilon(\omega,\omega,C,C)+\dots,$$
$$\mathrm{d}C+[\omega,C]_*=\Upsilon(\omega,C)+\Upsilon(\omega,C,C)+\dots$$

  where $\mathrm{d}=\mathrm{d}x^{\underline{m}}\partial_{\underline{m}}$ is space-time de Rham differential.
- Frame for reconstructing all HS vertices Υ is provided by Vasiliev equations.

Structure of 4d Vasiliev equations

- The idea is to introduce extra auxiliary $sp(4)$-variables $\boldsymbol{Z^A = \left(z^\alpha, \bar{z}^{\dot{\alpha}}\right)}$ and dual differentials $\boldsymbol{\theta^A}$ in a way that all Υ-s are encoded into evolution in (Z, θ)-directions.
- New master-fields are

$$\mathcal{W}(Z, Y; K|x) = W_{\underline{m}}(Z, Y; K|x)\mathrm{d}x^{\underline{m}} + S_A(Z, Y; K|x)\theta^A, \quad B(Z, Y; K|x).$$

- Physical fields – Z, θ-independent components:

$$\omega\,(Y; K|x) = W_{\underline{m}}\,(Z = 0, Y; K|x)\,\mathrm{d}x^{\underline{m}}, \quad C\,(Y; K|x) = B(Z = 0, Y; K|x).$$

- 4d Vasiliev equations [Vasiliev, Phys.Lett. B285 (1992) 225]

$$\begin{aligned} &\mathrm{d}\mathcal{W} + \mathcal{W} * \mathcal{W} = i\theta^A\theta_A + i\eta B * \gamma + i\bar{\eta} B * \bar{\gamma}\,, \\ &\mathrm{d}B + [\mathcal{W}, B]_* = 0\,. \end{aligned}$$

Structure of 4d Vasiliev equations

- 4d Vasiliev equations [Vasiliev, Phys.Lett. B285 (1992) 225]

$$\mathrm{d}\mathcal{W} + \mathcal{W} * \mathcal{W} = i\theta^A\theta_A + i\eta B * \gamma + i\bar{\eta} B * \bar{\gamma}\,,$$
$$\mathrm{d}B + [\mathcal{W}, B]_* = 0\,.$$

- HS algebra is realized on twistor variables via star-product

$$(F * G)(Z, Y) = \frac{1}{(2\pi)^4}\int dUdV\, F(Z+U, Y+U)\, G(Z-V, Y+V)\, e^{iU_A V^A},$$

which is generalization of universal enveloping of Weyl algebra:

$$[Y_A, Y_B]_* = -[Z_A, Z_B]_* = 2i\epsilon_{AB}, \quad [Y_A, Z_B]_* = 0.$$

- Inner Klein operators $\varkappa$ and $\bar{\varkappa}$: $\varkappa = e^{iz_\alpha y^\alpha}, \quad \{\varkappa, y_\alpha\}_* = \{\varkappa, z_\alpha\}_* = 0, \quad \varkappa * \varkappa = 1.$
- Outer Klein operators k and $\bar{k}$: $\{k, y_\alpha\} = \{k, z_\alpha\} = \{k, \theta_\alpha\} = 0, \quad k^2 = 1.$
- Central elements of the theory :

$$\delta^2(\theta) := \frac{1}{2}\theta^\alpha\theta_\alpha, \quad \gamma := 2k\varkappa\delta^2(\theta), \quad \delta^2(\bar{\theta}), \quad \bar{\gamma}.$$

Lorentz covariance

- Noncovariant equations are handy due to $\mathrm{d}^2 \equiv 0$.Though in AdS $\left(D^L\right)^2 \neq 0$, field redefinition to Lorentz-covariant formulation must exist in order to preserve equivalence principle.
- Lorentz-covariant derivative of a multispinor

$$D^L \phi_{\alpha(n),\dot{\alpha}(m)} = \mathrm{d}\phi_{\alpha(n),\dot{\alpha}(m)} - n\omega_\alpha{}^\beta \phi_{\beta\alpha(n-1),\dot{\alpha}(m)} - m\bar{\omega}_{\dot{\alpha}}{}^{\dot{\beta}} \phi_{\alpha(n),\dot{\beta}\dot{\alpha}(m-1)},$$

$$\left(D^L\right)^2 \phi_{\alpha(n),\dot{\alpha}(m)} = nR_\alpha{}^\beta \phi_{\beta\alpha(n-1),\dot{\alpha}(m)} - m\bar{R}_{\dot{\alpha}}{}^{\dot{\beta}} \phi_{\alpha(n),\dot{\beta}\dot{\alpha}(m-1)},$$

$$R_{\alpha\beta} = \mathrm{d}\omega_{\alpha\beta} - \omega_\alpha{}^\gamma \omega_{\gamma\beta}, \quad \bar{R}_{\dot{\alpha}\dot{\beta}} = \mathrm{d}\bar{\omega}_{\dot{\alpha}\dot{\beta}} - \bar{\omega}_{\dot{\alpha}}{}^{\dot{\gamma}} \bar{\omega}_{\dot{\gamma}\dot{\beta}}, \quad D^L R_{\alpha\beta} = D^L \bar{R}_{\dot{\alpha}\dot{\beta}} = 0.$$

- In HS master-fields all spinor indices are contracted with Z^A, Y^A or θ^A and we have

$$D^L F(Z;Y;\theta) = \left(\mathrm{d} + \omega^{AB}\left(Z_A \frac{\partial}{\partial Z^B} + Y_A \frac{\partial}{\partial Y^B} + \theta_A \frac{\partial}{\partial \theta^B}\right)\right) F(Z;Y;\theta).$$

- For θ-independent functions there is a star-product realization of connection part

$$\omega^{AB}\left(Z_A \frac{\partial}{\partial Z^B} + Y_A \frac{\partial}{\partial Y^B}\right) = \omega^{AB}\left[L_{AB}, f\right]_* ,$$

$$L_{AB} = -\frac{i}{4}\left(Y_A Y_B - Z_A Z_B\right).$$

Lorentz symmetry in θ-sector

θ-sectors of Vasiliev equations

$$[S_\alpha, S_\beta]_* = -2i\epsilon_{\alpha\beta}\left(1 + \eta B * \varkappa\right) ,$$
$$\{S_\alpha, B * \varkappa\}_* = 0 .$$

represents deformed oscillator algebra and respects $sp\,(2)$ for any B [Vasiliev, Int.J.Mod.Phys. A6 (1991) 1115]:

$$M_{\alpha\beta} = \frac{i}{8}\{S_\alpha, S_\beta\}_* ,$$

$$[M_{\alpha\alpha}, M_{\beta\beta}]_* = 2\epsilon_{\alpha\beta} M_{\alpha\beta}, \quad [M_{\alpha\alpha}, S_\beta]_* = \epsilon_{\alpha\beta} S_\alpha .$$

It can be checked that the proper field redefinition, bringing HS equations to Lorentz-covariant form, is [Vasiliev hep-th/9910096].

$$W \to W + \omega^{AB}\left(L_{AB} + M_{AB}\right) .$$

Field redefinitions

Now we move the other way around. Assume there exists a field redefinition to Lorentz-covariant system

$$W \to W' = W + \omega^{AB}(\ldots)$$

This should map

$$\begin{cases} \mathrm{d}S_\alpha + [W, S_\alpha]_* = 0\,, \\ \mathrm{d}B + [W, B]_* = 0\,, \end{cases} \longrightarrow \begin{cases} D^L S_\alpha + [W', S_\alpha]_* = 0\,, \\ D^L B + [W', B]_* = 0\,. \end{cases}$$

Consistency condition

$$\left(D^L\right)^2 S_\alpha = R^{\beta\gamma}\left[L_{\beta\gamma}, S_\alpha\right]_* - R_\alpha{}^\beta S_\beta$$

requires that

$$\left[D^L W' + W' * W' + R^{\beta\gamma}\left(L_{\beta\gamma} - \frac{i}{4} S_\beta * S_\gamma\right), S_\alpha\right]_* = 0.$$

Lorentz-covariant Vasiliev equations

Lorentz-covariant form of HS equations

$$D^L\mathcal{W}+\mathcal{W}*\mathcal{W}+R^{AB}\left(L_{AB}-\frac{i}{4}\frac{\partial}{\partial\theta^A}\mathcal{W}*\frac{\partial}{\partial\theta^B}\mathcal{W}\right)=i\theta^A\theta_A+i\eta B*\gamma+i\bar{\eta}B*\bar{\gamma}\,,$$

$$D^L B+[\mathcal{W},B]_*=0\,,$$

$$R^{AB}:=\mathrm{d}\omega^{AB}-\omega^{AC}\omega_C{}^B.$$

Field redefinition relating this to initial noncovariant form

$$\mathcal{W}'=\mathcal{W}-\omega^{AB}\left(L_{AB}-\frac{i}{4}\frac{\partial}{\partial\theta^A}\mathcal{W}*\frac{\partial}{\partial\theta^B}\mathcal{W}\right).$$

Extra Stueckelberg symmetry gauging away Lorentz connection [Iazeolla, Sundell, JHEP 1112 (2011) 084]

$$\delta_\xi\omega_{AB}=\xi_{AB},\quad \delta_\xi\mathcal{W}'=-\xi^{AB}\left(L_{AB}-\frac{i}{4}\frac{\partial}{\partial\theta^A}\mathcal{W}*\frac{\partial}{\partial\theta^B}\mathcal{W}\right),\quad \delta_\xi B=0.$$

To fix ω_{AB} we demand $\mathcal{W}$ to contain no connection-type terms

$$\frac{\partial^2}{\partial y^\alpha\partial y^\beta}\mathcal{W}'|_{Z,Y=0}=\frac{\partial^2}{\partial\bar{y}^{\dot\alpha}\partial\bar{y}^{\dot\beta}}\mathcal{W}'|_{Z,Y=0}=0.$$

Extended HS equations

In [Vasiliev, Nucl.Phys. B916 (2017) 219] extension of $4d$ HS equations was proposed, which includes higher forms and closed space-time functionals of HS fields (BH charges [Didenko, NM, Vasiliev, JHEP 1703 (2017) 164] and AdS/CFT functional presumably)

$$\mathbb{W} = \mathcal{W} + \mathcal{W}_3, \quad \mathbb{B} = B + B_2, \quad \mathcal{L}(x) := \mathcal{L}_2(x) + \mathcal{L}_4(x) .$$

Extended system has the form

$$\begin{aligned}
&\mathrm{d}\mathbb{W} + \mathbb{W} * \mathbb{W} = i\theta^A\theta_A + i\eta\mathbb{B} * \gamma + i\bar{\eta}\mathbb{B} * \bar{\gamma} + ig\gamma * \bar{\gamma} + \mathcal{L}(x) \ , \\
&\mathrm{d}\mathbb{B} + [\mathbb{W}, \mathbb{B}]_* = 0 , \\
&\mathrm{d}\mathcal{L}(x) = 0 ,
\end{aligned}$$

Higher forms do not influence sector of 1- and 0-forms and are expressed via them. The system possesses enhanced gauge symmetry

$$\begin{aligned}
&\delta\mathbb{W} = \mathrm{d}\epsilon + [\mathbb{W}, \epsilon]_* + i\eta\xi * \gamma + i\bar{\eta}\xi * \bar{\gamma} + \zeta , \\
&\delta\mathbb{B} = \mathrm{d}\xi + \{\mathbb{W}, \xi\}_* + [\mathbb{B}, \epsilon]_* \ , \\
&\delta\mathcal{L} = \mathrm{d}\zeta(x) \ .
\end{aligned}$$

Twisted sector and deformed oscillators

We expand twistor fields into components

$$\mathbb{W}\Big|_{dx=0} = S_A\theta^A + 2t_\alpha\theta^\alpha\delta^2\left(\bar\theta\right) + 2\bar t_{\dot\alpha}\bar\theta^{\dot\alpha}\delta^2\left(\theta\right) ,$$

$$\mathbb{B}\Big|_{dx=0} = B + 2b\delta^2\left(\theta\right) + 2\bar b\delta^2\left(\bar\theta\right) + b_{\alpha\dot\alpha}\theta^\alpha\bar\theta^{\dot\alpha} .$$

Then HS equations take the form

$$\left[S_\alpha, S_\beta\right]_* = -2i\epsilon_{\alpha\beta}\left(1+\eta B*\varkappa\right) , \quad \left[S_\alpha, \bar S_{\dot\alpha}\right]_* = 0 ,$$

$$\left[S_\alpha, \bar b\right]_* + \left[t_\alpha, B\right]_* = 0 , \quad \left[S_\alpha, B\right]_* = 0 , \quad \frac{1}{2}\left[S_\alpha, b^\alpha{}_{\dot\alpha}\right]_* + \left[\bar t_{\dot\alpha}, B\right]_* = 0 ,$$

$$\left[t_\alpha, S^\alpha\right]_* + \left[\bar t_{\dot\alpha}, \bar S^{\dot\alpha}\right]_* = -2i\left(\eta\bar b*\varkappa + \bar\eta b*\bar\varkappa + g\varkappa*\bar\varkappa\right) .$$

It turns out that this system admits realization of Lorentz symmetry as gauge transformation. For $\mathbb{B}=0$ case gauge symmetries are

$$\delta_\Lambda S_\alpha = \left[S_\alpha, \epsilon\right]_* ,$$

$$\delta_\Lambda t_\alpha = \left[t_\alpha, \epsilon\right]_* + \left[\phi, S_\alpha\right] + \left[\psi_\alpha{}^{\dot\beta}, \bar S_{\dot\beta}\right] ,$$

$$\delta_\Lambda \bar t_{\dot\alpha} = \left[\bar t_{\dot\alpha}, \epsilon\right]_* + \left[\bar\phi, \bar S_{\dot\alpha}\right] - \left[\bar\psi^\beta{}_{\dot\alpha}, S_\beta\right] .$$

Then choosing

$$\epsilon = -\frac{i}{4}\Lambda^{\alpha\beta}S_\alpha * S_\beta , \quad \phi = \frac{i}{4}\Lambda^{\alpha\beta}\left\{S_\alpha * t_\beta\right\} , \quad \psi_{\alpha\dot\alpha} = -\frac{i}{4}\Lambda_\alpha{}^\beta\left\{S_\beta, \bar t_{\dot\alpha}\right\}_*$$

we get

$$\delta_\Lambda S_\alpha = \Lambda_\alpha{}^\beta S_\beta , \quad \delta_\Lambda t_\alpha = \Lambda_\alpha{}^\beta t_\beta .$$

Generalized deformed oscillator algebra

Oscillator (Weyl) algebra:

$$[y_\alpha, y_\beta] = -2i\epsilon_{\alpha\beta}.$$

Deformed oscillator algebra:

$$[y_\alpha, y_\beta] = -2i\epsilon_{\alpha\beta}(1+\nu K), \quad \{y_\alpha, K\} = 0, \quad KK = 1.$$

For any value of central element ν y-bilinears form $sp(2)$

$$M_{\alpha\beta} = \{y_\alpha, y_\beta\}, \quad [M_{\alpha\alpha}, M_{\beta\beta}] \sim \epsilon_{\alpha\beta} M_{\alpha\beta}.$$

Adding another copy $\bar{y}$ generate full Lorentz algebra $sp(2|\mathbb{C})$. Extended equations provide a nontrivial generalization of deformed oscillators that still respects $sp(2|\mathbb{C})$. For $B = const$, $b = 0$ it is

$$[S_\alpha, S_\beta] = -2i\epsilon_{\alpha\beta}(1+\nu K), \qquad [\bar{S}_{\dot\alpha}, \bar{S}_{\dot\beta}] = -2i\epsilon_{\dot\alpha\dot\beta}(1+\nu\bar{K}),$$

$$\{S_\alpha, K\} = 0, \qquad \{\bar{S}_{\dot\alpha}, \bar{K}\} = 0,$$

$$\{t_\alpha, K\} = 0, \qquad \{\bar{t}_{\dot\alpha}, \bar{K}\} = 0,$$

$$[S_\alpha, \bar{S}_{\dot\alpha}] = 0, \qquad [t_\alpha, S^\alpha] + [\bar{t}_{\dot\alpha}, \bar{S}^{\dot\alpha}] = 2igK\bar{K}.$$

Lorentz-covariant ext-HS system

Once again, assuming the existence of field redefinition and analyzing consistency condition in twisted sector, we get Lorentz-covariant form of extended HS equations

$$D^L\mathbb{W}+\mathbb{W}*\mathbb{W}+R^{AB}\left(L_{AB}-\frac{i}{4}\frac{\partial}{\partial\theta^A}\mathcal{W}*\frac{\partial}{\partial\theta^B}\mathcal{W}\right)=i\theta^A\theta_A+i\eta\mathbb{B}*\gamma+i\bar{\eta}\mathbb{B}*\bar{\gamma}+ig\gamma*\bar{\gamma}+\mathcal{L}-$$
$$-\left(\frac{\eta}{4}R^{\alpha\beta}\frac{\partial}{\partial\theta^\alpha}\mathbb{B}*\frac{\partial}{\partial\theta^\beta}\gamma-\frac{i\eta}{32}R^{\alpha\alpha}R^{\beta\beta}\frac{\partial^2}{\partial\theta^\alpha\partial\theta^\beta}\mathbb{B}*\frac{\partial^2}{\partial\theta^\alpha\partial\theta^\beta}\gamma+c.c.\right)$$
$$D^L\mathbb{B}+[\mathbb{W},\mathbb{B}]_*-\frac{i}{4}R^{AB}\left\{\frac{\partial}{\partial\theta^A}\mathbb{B},\frac{\partial}{\partial\theta^B}\mathbb{W}\right\}_*=0\,.$$

Stueckelberg symmetry switching between covariant and noncovariant systems

$$\delta_\xi\omega_{AB}=\xi_{AB},\quad \delta_\xi\mathbb{B}=\frac{i}{4}\xi^{AB}\left\{\frac{\partial}{\partial\theta^A}\mathbb{B},\frac{\partial}{\partial\theta^B}\mathbb{W}\right\}_*$$
$$\delta_\xi\mathbb{W}=-\xi^{\alpha\beta}\left(L_{\alpha\beta}-\frac{i}{4}\frac{\partial}{\partial\theta^\alpha}\mathbb{W}*\frac{\partial}{\partial\theta^\beta}\mathbb{W}\right)-\frac{\eta}{4}\xi^{\alpha\beta}\frac{\partial}{\partial\theta^\alpha}\mathbb{B}*\frac{\partial}{\partial\theta^\beta}\gamma+$$
$$+\frac{i\eta}{32}\left(\xi^{\alpha\alpha}R^{\beta\beta}+\xi^{\beta\beta}R^{\alpha\alpha}\right)\frac{\partial^2}{\partial\theta^\alpha\partial\theta^\beta}\mathbb{B}*\frac{\partial^2}{\partial\theta^\alpha\partial\theta^\beta}\gamma+c.c.$$

Lorentz symmetry and central elements

- Consider noncovariant non-extended equation

$$\mathrm{d}\mathcal{W} + \mathcal{W} * \mathcal{W} = i\theta^A\theta_A + i\eta B * \gamma + i\bar{\eta} B * \bar{\gamma}\,.$$

Rescaling properly θ and master-fields one can set central element to be $i\nu\theta^A\theta_A$.
Covariantized equation becomes

$$D^L\mathcal{W} + \mathcal{W} * \mathcal{W} + R^{AB}\left(L_{AB} - \frac{i}{4\nu}\frac{\partial}{\partial\theta^A}\mathcal{W} * \frac{\partial}{\partial\theta^B}\mathcal{W}\right) = i\theta^A\theta_A + i\eta B * \gamma + i\bar{\eta} B * \bar{\gamma}\,.$$

So limit $\nu \to 0$ is forbidden by requirement of Lorentz symmetry.

- Consider Lorentz-covariant extended equation with $ig\gamma * \bar{\gamma}$ replaced by general 4-form central element c.

$$D^L\mathbb{W} + \mathbb{W} * \mathbb{W} + R^{AB}\left(L_{AB} - \frac{i}{4}\frac{\partial}{\partial\theta^A}\mathcal{W} * \frac{\partial}{\partial\theta^B}\mathcal{W}\right) = i\theta^A\theta_A + i\eta\mathbb{B} * \gamma + i\bar{\eta}\mathbb{B} * \bar{\gamma} + c + \mathcal{L} -$$

$$-\left(\frac{\eta}{4}R^{\alpha\beta}\frac{\partial}{\partial\theta^\alpha}\mathbb{B} * \frac{\partial}{\partial\theta^\beta}\gamma - \frac{i\eta}{32}R^{\alpha\alpha}R^{\beta\beta}\frac{\partial^2}{\partial\theta^\alpha\partial\theta^\beta}\mathbb{B} * \frac{\partial^2}{\partial\theta^\alpha\partial\theta^\beta}\gamma + c.c.\right)$$

Consistency in particular requires

$$R^{AB}\left\{\frac{\partial}{\partial\theta^A}c, \frac{\partial}{\partial\theta^B}\mathbb{W}\right\} = 0$$

which holds for $c = ig\gamma * \bar{\gamma}$ but not for other central elements (like $\delta^4(\theta)$).

Conclusions

- We show that extended HS equations admit local Lorentz symmetry.
- We find that requirement of Lorentz symmetry restricts the form of central elements entering HS equations.
- We find that extended HS equations provide a nontrivial generalization of deformed oscillator algebra.
- We find that operator of HS perturbation theory are drastically simplified in Lorentz-covariant formulation (stayed outside the talk, but that's really true!)