Non-Perturbative Bundle Moduli Superpotentials in Heterotic String Theory

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Also- Buchbinder, Lukas, Ovrut, Ruehle
Buchbinder, Ling, Ovrut

The Instanton Superpotential:

Let X be a Calabi-Yau threefold and C be a holomorphic,isolated, genus zero curve in X. The general form of the superpotential induced by a string wrapping C is

$$W(C) = \exp\Big[-\frac{A(C)}{2\pi\alpha'} + i\int_C B\Big] \frac{\mathrm{Pfaff}(\bar{\partial}_{V_{C(-1)}})}{[\det'(\bar{\partial}_{\mathcal{O}})]^2 \det(\bar{\partial}_{NC})}$$

where A(C) is the area of the curve given by

$$A(C) = \int_C \omega$$

ω is the Kahler form on X and **B** is the antisymmetric heterotic two-form.

V is the holomorphic vector bundle on X, $\mathcal{O}_C(-1)$ is the spin bundle on C and we define

$$V_C(-1) = V|_C \otimes \mathcal{O}_C(-1)$$

Then $\mathrm{Pfaff}(\bar{\partial}_{V_{C(-1)}})$ is the Pfaffian of the Dirac operator with gauge connection in V restricted to C. $\Rightarrow$ $\mathrm{Pfaff}(\bar{\partial}_{V_{C(-1)}})$ is a homogeneous polynomial in the vector bundle moduli associated with V at curve C.

$\det(\bar{\partial}_{NC})$ depends on the complex structure moduli. We, henceforth, ignore the constant $[\det'(\bar{\partial}_{\mathcal{O}})]^2$ and fix the complex structure moduli.

In general, a given homology class of X contains more than one homogeneous, isolated, genus zero curve. The number of these curves is called the Gromov-Witten invariant. All such curves have the same area, the same classical action and the same exponential prefactor. However, the one-loop determinants which determine the Pfaffian and $\det(\bar{\partial}_{NC})$ are, in general, different. $\Rightarrow$ the superpotential from all such curves in the homology class [C] of curve C is

$$W([C]) = \exp\Big[-\frac{A(C)}{2\pi\alpha'} + i\int_C B\Big] \sum_{i=1}^{n_{[C]}} \frac{\mathrm{Pfaff}(\bar{\partial}_{V_{C_i}(-1)})}{[\det\bar{\partial}_{\mathcal{O}_{C_i}(-1)}]^2}$$

where $n_{[C]}$ is the Gromov-Witten invariant of [C]. Therefore the complete superpotential on X is

$$W = \sum_{[C]\in H_2} W([C])$$

The Beasley-Witten Theorem:

Let $\tilde{X}$ be a CICY threefold in a product of projective spaces $\mathcal{A} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_a}$.

That is, it is defined by polynomial equations

$$p_1 = 0, \ldots, p_m = 0 \text{ where } \sum_{i=1}^{a} n_i - m = 3.$$

Furthermore, assume that

favorable ⟶ $\omega_{\tilde{X}} = \omega_{\mathcal{A}}|_{\tilde{X}}$ and $\tilde{V} = \mathcal{V}|_{\tilde{X}}$

Then the Beasley-Witten theorem $\Rightarrow$ for any homology class [C]

$$W([C]) = \exp\Big[-\frac{A(C)}{2\pi\alpha'} + i\int_C B\Big] \sum_{i=1}^{n_{[C]}} \frac{\mathrm{Pfaff}(\bar{\partial}_{V_{C_i}(-1)})}{[\det \bar{\partial}_{\mathcal{O}_{C_i}(-1)}]^2} = 0$$

and, hence

$$W = \sum_{[C]\in H_2} W([C]) = 0$$

This $\Rightarrow$ that such vacua can never develop a potential for the vector bundle moduli. A big problem! Is it possible to get around this? YES!

Quotient Threefolds with Torsion:

Consider a CICY $\tilde{X}$ that admits a freely acting finite isometry Γ. Construct the quotient threefold

$$X = \frac{\tilde{X}}{\Gamma}$$

Such manifolds can, and often do, have "discrete torsion". That is

$$H_2(\mathrm{X}, \mathbb{Z}) = \mathbb{Z}^k \oplus \mathrm{G}_{tor}\,, \quad k > 0$$

$\mathbf{G}_{tor}$ is the finite torsion group with r generators $\beta_1, \ldots, \beta_r$.

Consider any holomorphic, isolated, genus zero curve C_i in [C].

This can be associated with the $\mathbf{G}_{tor}$ group character

$$\prod_{\alpha=1}^{r} \chi_\alpha^{\beta_\alpha(C_i)}$$

It follows that the complete instanton superpotential associated with [C] is now

$$W([C]) = \exp\Big[-\frac{A(C)}{2\pi\alpha'} + i\int_C B\Big] \sum_{i=1}^{n_{[C]}} \frac{\mathrm{Pfaff}(\bar{\partial}_{V_{C_i}(-1)})}{[\det\bar{\partial}_{\mathcal{O}_{C_i}(-1)}]^2} \prod_{\alpha=1}^{r} \chi_\alpha^{\beta_\alpha(C_i)}$$

$X = \frac{\tilde{X}}{\Gamma}$ is no longer a CICY of the ambient space $\mathcal{A} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_a}$,
nor is

$$\omega_X = \omega_{\mathcal{A}}\big|_X \quad \text{and} \quad V = \mathcal{V}\big|_X$$

generically true. Hence,the Beasley-Witten theorem no longer applies

and it is possible that

$$W([C]) = \exp\Big[-\frac{A(C)}{2\pi\alpha'} + i\int_C B\Big] \sum_{i=1}^{n_{[C]}} \frac{\text{Pfaff}(\bar{\partial}_{V_{C_i}(-1)})}{[\det\bar{\partial}_{\mathcal{O}_{C_i}(-1)}]^2} \prod_{\alpha=1}^{r} \chi_\alpha^{\beta_\alpha(C_i)} \neq 0$$

and, hence,

$$W = \sum_{[C]\in H_2} W([C]) \neq 0$$

We now show in a physically relevant example that this is indeed the case!

A Schoen Threefold:

Consider the ambient space

$$\mathcal{A} = \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$$

with homogeneous coordinates

$$([t_0 : t_1], [x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \in \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$$

Define the CICY space $\tilde{X}$ using the two polynomial equations

$$\begin{aligned} F_1 &= t_0(x_0^3 + x_1^3 + x_2^3) + t_1(x_0 x_1 x_2) = 0\,, \\ F_2 &= (\lambda_1 t_0 + t_1)(y_0^3 + y_1^3 + y_2^3) + (\lambda_2 t_0 + \lambda_3 t_1)(y_0 y_1 y_2) = 0 \end{aligned}$$

This threefold is self mirror with $h^{1,1} = h^{2,1} = 19$. Note that

unfavorable ⟶ $$h^{1,1}(\tilde{X}) > h^{1,1}(\mathcal{A}) = 3$$

which already violates the Beasley-Witten assumption that $\omega_{\tilde{X}} = \omega_{\mathcal{A}}|_{\tilde{X}}$.

However, this aspect of Beasley-Witten violation is hard to use to compute the instanton potential.

$\star$ However, see Buchbinder, Lukas, Ovrut, Ruehle conjecture that unfavorable $\Rightarrow$ $W[C] \neq 0$

Each polynomial equation defines a rational elliptic surface

$$dP_9 \in \mathbb{P}^1 \times \mathbb{P}^2$$

$\Longrightarrow$ $\tilde{X}$ is a double elliptic fibration over $\mathbb{P}^1$ That is,

$$\begin{array}{lccc}
\dim_{\mathbb{C}} = 3: & & \tilde{X} & \\
& \swarrow^{\pi_1} & & \searrow^{\pi_2} \\
\dim_{\mathbb{C}} = 2: & dP_9 = B_1 & & B_2 = dP_9 \\
& \searrow & & \swarrow \\
\dim_{\mathbb{C}} = 1: & & \mathbb{P}^1 &
\end{array}$$

Note that $F_1 = F_2 = 0$ are invariant under the actions

$$g_1: \begin{cases} [x_0 : x_1 : x_2] \mapsto [x_0 : \zeta x_1 : \zeta^2 x_2] \\ [t_0 : t_1] \mapsto [t_0 : t_1] \text{ (no action)} \\ [y_0 : y_1 : y_2] \mapsto [y_0 : \zeta y_1 : \zeta^2 y_2] \end{cases}$$

$$g_2: \begin{cases} [x_0 : x_1 : x_2] \mapsto [x_1 : x_2 : x_0] \\ [t_0 : t_1] \mapsto [t_0 : t_1] \text{ (no action)} \\ [y_0 : y_1 : y_2] \mapsto [y_1 : y_2 : y_0], \end{cases}$$

where $\zeta = e^{2\pi i/3}$. That is, $\tilde{X}$ has the finite isometry group

$$\Gamma = \mathbb{Z}_3 \times \mathbb{Z}_3$$

Note that $\tilde{X}$ is a specific example of a so-called Schoen threefold.

We can now define the quotient threefold

$$X = \tilde{X}/(\mathbb{Z}_3 \times \mathbb{Z}_3)$$

Again, this threefold is self mirror, but now with $h^{1,1} = h^{2,1} = 3$. The second homology group is found to be

$$H_2(X, \mathbb{Z}) = \mathbb{Z}^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$$

$\Rightarrow$

$$G_{tor} = \mathbb{Z}_3 \oplus \mathbb{Z}_3$$

As discussed above, $X = \dfrac{\tilde{X}}{\Gamma}$ is no longer a CICY of the ambient space $\mathcal{A} = \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$ nor are ,

$$\omega_X = \omega_{\mathcal{A}}\big|_X \quad \text{and} \quad V = \mathcal{V}\big|_X$$

necessarily true. Hence, the Beasley-Witten theorem no longer applies.

What are the classes in the second homology group on X?

Recall that $H_2(X,\mathbb{Z}) = \mathbb{Z}^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$. Label the generators of $\mathbb{Z}^3$ as

$$\mathrm{C}_1,\, C_2,\, C_3$$

where

$$\int_{C_i} \omega_j = \delta_{ij}$$

respectively, and the generators of $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ as b_1 and b_2 where

$$b_1^3 = b_2^3 = 1$$

Then, any class of the second cohomology group of X can be written as

$$[C] = (n_1, n_2, n_3, m_1, m_2) \in H_2(X,\mathbb{Z}) = \mathbb{Z}^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$$

where $n_1, n_2, n_3 \in \mathbb{Z}$ and $m_1, m_2 = 0, 1, 2$. Can one compute the Gromov-Witten invariants in each such cohomology class?

Yes, using the mirror symmetry of the quotient threefold X.

Taking $n_1 = 1$, we find that

$n_2 \backslash n_3$	0	1	2	3	4	5
0	1	4	14	40	105	252
1	4	16	56	160	420	1008
2	14	56	196	560	1470	3528
3	40	160	560	1600	4200	10080
4	105	420	1470	4200	11025	26460
5	252	1008	3528	10080	26460	63504

Table: instanton numbers $n_{(1,n_2,n_3,m_1,m_2)}$ for arbitrary m_1, m_2.

Note that each of the $(1,0,0,m_1,m_2)$ classes has only a single homogeneous, isolated, genus zero curve and, hence, the instanton superpotential in each such class cannot cancel via the Beasley-Witten theorem. If there had had been vanishing torsion on the quotient X, then the class (1,0,0) would have contained 9 curves. These might have canceled against each other.

We now compute the instanton superpotential for each $(1,0,0,m_1,m_2)$ class.

To do this, we must have an explicit representation of these curves. To begin, consider the 9 curves in $H_2(X,\mathbb{R})$; that is, $H_2(X,\mathbb{Z})$ ignoring torsion.

The pre-image of these in $\tilde{X}$ are 81 holomorphic, isolated, genus zero curves.

These arise as $\mathbb{P}^1 \times$ the 9 X 9=81 points solving the equations

$$x_0x_1x_2 = 0\,, \quad x_0^3 + x_1^3 + x_2^3 = 0\,, \quad y_0y_1y_2 = 0\,, \quad y_0^3 + y_1^3 + y_2^3 = 0$$

on $\mathbb{P}^2 \times \mathbb{P}^2$. Since these 81 points are distinct, it follows that these curves are indeed isolated. Due to the $\mathbb{Z}_3 \times \mathbb{Z}_3$ symmetry, these 81 curves split into 9 orbits under the action of $\mathbb{Z}_3 \times \mathbb{Z}_3$ — each orbit containing 9 curves. When one descends to the quotient space X, all curves in one orbit become a single isolated curve.

Hence, one obtains the 9 curves in $H_2(X,\mathbb{R})$ which split into the 9 different torsion classes $(1,0,0,m_1,m_2)$.

To explicitly compute the superpotential of due to these curves, it is essential that we have an explicit representation of one curve in each of the nine $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbits in $\tilde{X}$.

To do this first consider the representations

$$\zeta = e^{2\pi i/3} \longrightarrow \quad g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & \xi^2 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

of the two generators of $\mathbb{Z}_3 \times \mathbb{Z}_3$ acting on $[x_0 : x_1 : x_2]$ and $[y_0 : y_1 : y_2]$.

Combine these spaces into a six-vector, and consider the solution point

$$s_1 = (1, -1, 0, 1, -1, 0)^T$$

It corresponds to the curve

$$C_1 = \mathbb{P}^1 \times s_1 = [t_0 : t_1] \times [1 : -1 : 0] \times [1 : -1 : 0] \subset \tilde{X} \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$$

We now construct the remaining 8 curves $C_i = \mathbb{P}^1 \times s_i,\ i = 2, \ldots, 9$ as

$$s_2 = \begin{pmatrix} g_1 & 0 \\ 0 & 1 \end{pmatrix} s_1, \quad s_3 = \begin{pmatrix} 1 & 0 \\ 0 & g_1 \end{pmatrix} s_1, \quad s_4 = \begin{pmatrix} g_2 & 0 \\ 0 & 1 \end{pmatrix} s_1, \quad s_5 = \begin{pmatrix} 1 & 0 \\ 0 & g_2 \end{pmatrix} s_1,$$

$$s_6 = \begin{pmatrix} g_1 g_2 & 0 \\ 0 & 1 \end{pmatrix} s_1, \quad s_7 = \begin{pmatrix} 1 & 0 \\ 0 & g_1 g_2 \end{pmatrix} s_1, \quad s_8 = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} s_1, \quad s_9 = \begin{pmatrix} g_2 & 0 \\ 0 & g_1 \end{pmatrix} s_1$$

The curves cannot be obtained from one another by the action of $\mathbb{Z}_3 \times \mathbb{Z}_3$ and, hence, each defines one of the 9 orbits on $\tilde{X}$.

The Vector Bundle:

The vector bundle $\tilde{V}$ on $\tilde{X}$ will be defined by "extension" from 3 line bundles

L_1, L_2, L_3 on $\tilde{X}$ satisfying the property that

$$L_1 \otimes L_2 \otimes L_3 = \mathcal{O}_{\tilde{X}}$$

Define $\tilde{V}$ as the sequence of extensions

$$0 \longrightarrow L_1 \longrightarrow \tilde{W} \longrightarrow L_2 \longrightarrow 0$$
$$0 \longrightarrow \tilde{W} \longrightarrow \tilde{V} \longrightarrow L_3 \longrightarrow 0$$

Explicitly, we will assume that

$$L_1 = \mathcal{O}_{\tilde{X}}(-2\phi + 2\tau_1 + \tau_2)\,,$$
$$L_2 = \mathcal{O}_{\tilde{X}}(\tau_1 - \tau_2)\,,$$
$$L_3 = \mathcal{O}_{\tilde{X}}(2\phi - 3\tau_1)$$

$\phi,\ \tau_1,\ \tau_2$ are the $\mathbb{Z}_3 \times \mathbb{Z}_3$ invariant divisors on $\tilde{X}$

For $\tilde{V}$ to have structure group SU(3), it must have a non-trivial space of extensions

$$H^1(\tilde{X}, L_1 \otimes L_2^*) \quad \text{and} \quad H^1(\tilde{X}, \tilde{W} \otimes L_3^*)$$

We find that

$$h^1(\tilde{X}, L_1 \otimes L_2^*) = 18 \quad , \quad h^1(\tilde{X}, \tilde{W} \otimes L_3^*) = 117$$

Finally, the "moduli space" of $\tilde{V}$ is given by

$$\mathcal{M}(\tilde{V}) = \mathbb{P}H^1(\tilde{X}, L_1 \otimes L_2^*) + \mathbb{P}H^1(\tilde{X}, \tilde{W} \otimes L_3^*)$$

and, hence,

$$\dim\mathcal{M}(\tilde{V}) = (h^1(\tilde{X}, L_1 \otimes L_2^*) - 1) + (h^1(\tilde{X}, \tilde{W} \otimes L_3^*) - 1)$$

$$= 17 + 116 = 133$$

Note that if we consider the quotient vector bundle V on X, then

$$h^1(X, L_1 \otimes L_2^*) = 18/9 = 2 \ , \quad h^1(X, W \otimes L_3^*) = 117/9 = 13$$

It follows that

$$\dim\mathcal{M}(V) = 1 + 12 = 13$$

and, hence, there are 13 vector bundle moduli on the quotient space.

Does the vector bundle $\tilde{V}$ on $\tilde{X}$ descend from a bundle $\tilde{\mathcal{V}}$ on the ambient space $\mathcal{A}$? Yes. One simply carries out the identical construction using

$$\mathcal{L}_1 = \mathcal{O}_{\mathcal{A}}(-2, 2, 1)\,, \quad \mathcal{L}_2 = \mathcal{O}_{\mathcal{A}}(0, 1, -1)\,, \quad \mathcal{L}_3 = \mathcal{O}_{\mathcal{A}}(2, -3, 0)$$

Perhaps not surprisingly, on can show that

$$H^1(\tilde{X}, L_1 \otimes L_2^*) = H^1(\mathcal{A}, \mathcal{L}_1 \otimes \mathcal{L}_2^*) = H^1(\mathcal{A}, \mathcal{O}_{\mathcal{A}}(-2,1,2))$$

and also that

$$H^1(\tilde{X}, \tilde{W} \otimes L_3^*) = \frac{H^1(\mathcal{A}, \tilde{\mathcal{W}} \otimes \mathcal{L}_3^*)}{F_1 \cdot H^1(\mathcal{A}, \tilde{\mathcal{W}} \otimes \mathcal{N}^* \otimes \mathcal{L}_3^*)} = \frac{H^1(\mathcal{A}, \mathcal{O}_{\mathcal{A}}(-4,5,1))}{F_1 \cdot H^1(\mathcal{A}, \mathcal{O}_{\mathcal{A}}(-5,2,1))}$$

The ambient space description allows us to parameterize the moduli that descend to V on X. From the Kunneth and Bott formulas, it follows that

$$H^1(\mathcal{A}, \mathcal{O}_{\mathcal{A}}(-4,5,1)) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-4)) \otimes H^0(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(5,1))$$

Note that

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-4)) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))^*$$

is 3-dimensional with a natural basis $\{r_0^2, r_0 r_1, r_1^2\}$ dual to the basis $\{t_0^2, t_0 t_1, t_1^2\}$ of degree 2 polynomials on $\mathbb{P}^1$. Hence, any element $v \in H^1(\mathcal{A}, \mathcal{O}_{\mathcal{A}}(-4,5,1))$ can be written as

$$v = r_0^2 f_1(\mathbf{x}, \mathbf{y}) + r_0 r_1 f_2(\mathbf{x}, \mathbf{y}) + r_1^2 f_3(\mathbf{x}, \mathbf{y})$$

where f_1, f_2, f_3 are homogeneous polynomials of degree (5,1) on $\mathbb{P}^2 \times \mathbb{P}^2$.

The 189 coefficients in the polynomials f_1, f_2, f_3 can be viewed as the coordinates, that is, the moduli, on $H^1(\mathcal{A}, \mathcal{O}_{\mathcal{A}}(-4,5,1))$. Modding out by $F_1 \cdot H^1(\mathcal{A}, \mathcal{O}_{\mathcal{A}}(-5,2,1))$

$\Longrightarrow$ dim $\mathbb{P}H^1(\tilde{X}, \tilde{W} \otimes L_3^*) = 116$, as previously. Let us now restrict to polynomials that are invariant under $\mathbb{Z}_3 \times \mathbb{Z}_3$. A basis for such polynomials is given by

$$
\begin{aligned}
E_1 &= x_0^5 y_0 + x_1^5 y_1 + x_2^5 y_2 \,, \\
E_2 &= x_0^2 x_1^3 y_0 + x_1^2 x_2^3 y_1 + x_2^2 x_0^3 y_2 \,, \\
E_3 &= x_0^2 x_2^3 y_0 + x_1^2 x_0^3 y_1 + x_2^2 x_1^3 y_2 \,, \\
E_4 &= x_0^2 x_1 x_2 y_0 + x_1^2 x_2 x_0 y_1 + x_2^2 x_0 x_1 y_2 \,,
\end{aligned}
\qquad
\begin{aligned}
E_5 &= x_1^4 x_2 y_0 + x_2^4 x_0 y_1 + x_0^4 x_1 y_2 \,, \\
E_6 &= x_0^5 y_0 + x_1^5 y_1 + x_2^5 y_2 \,, \\
E_7 &= x_1 x_2^4 y_0 + x_2 x_0^4 y_1 + x_0 x_1^4 y_2 \,.
\end{aligned}
$$

The invariant polynomials are then given by

$$
f_1 = \sum_{\alpha=1}^{7} a_\alpha E_\alpha \,, \qquad f_2 = \sum_{\alpha=1}^{7} b_\alpha E_\alpha \,, \qquad f_3 = \sum_{\alpha=1}^{7} c_\alpha E_\alpha
$$

Note that there are 21 coefficients $(a_\alpha, b_\alpha, c_\alpha)$. However, one must mod out

$$
F_1 \cdot H^1(\mathcal{A}, \mathcal{O}_{\mathcal{A}}(-5,2,1))
$$

in $H^1(\tilde{X}, \tilde{W} \otimes L_3^*)$. The result is the constraints

$$
\begin{aligned}
&a_1 + a_2 + a_3 = 0\,, \quad a_4 + a_5 + a_6 = 0\,, \\
&a_4 + b_1 + b_2 + b_3 = 0\,, \quad a_7 + b_4 + b_5 + b_6 = 0\,, \\
&b_4 + c_1 + c_2 + c_3 = 0\,, \quad b_7 + c_4 + c_5 + c_6 = 0\,, \\
&c_4 = 0\,, \qquad c_7 = 0\,.
\end{aligned}
$$

We can choose the 13 coordinates

$$a_1, a_2, a_5, b_1, b_2, b_3, b_5, b_6, c_1, c_2, c_3, c_5, c_6$$

as independent parameters. It follows that

dim $\mathbb{P}H^1(X, W \otimes L_3^*)$ = 13-1= 12

That is, there are 12 moduli of this type on V and they are parameterized by $a_1, a_2, a_5, b_1, b_2, b_3, b_5, b_6, c_1, c_2, c_3, c_5, c_6$ as projective coordinates.

Similarly, we can show that

dim $\mathbb{P}H^1(X, L_1 \otimes L_2^*)$ = 2-1= 1

However, this modulus does not appear in the Pfaffians and we will ignore it.

Computation of the Pfaffians:

In the following, I will simply state the results of our calculations. First, for an arbitrary homogeneous, isolated, genus zero curve we find

$$\mathrm{Pfaff}_{\tilde{X}}(\bar{\partial}_{\tilde{V}_{C(-1)}}) \sim (f_1 f_3 - f_2^2)(\mathbf{x},\mathbf{y}) = \sum_{\alpha,\beta=1}^{7} (a_\alpha c_\beta - b_\alpha b_\beta) E_\alpha E_\beta(\mathbf{x},\mathbf{y})$$

Applying this to our nine curves $s_i, i = 1, \ldots, 9$ and denoting

$$\mathcal{R}_{\tilde{X},i} = (f_1 f_3 - f_2^2)(s_i)\,, \qquad i = 1, \ldots, 9$$

we find that

$$\begin{aligned}
\mathcal{R}_{\tilde{X},1} &= -(2b_1 - b_2 - b_3)^2 + (2a_1 - a_2 - a_3)(2c_1 - c_2 - c_3)\,,\\
\mathcal{R}_{\tilde{X},2} &= -(b_2 + b_3\zeta^2 + b_1\zeta)^2 + (a_2 + a_3\zeta^2 + a_1\zeta)(c_2 + c_3\zeta^2 + c_1\zeta)\,,\\
\mathcal{R}_{\tilde{X},3} &= -(b_2 + b_3\zeta + b_1\zeta^2)^2 + (a_2 + a_3\zeta + a_1\zeta^2)(c_2 + c_3\zeta^2 + c_1\zeta)\,,\\
\mathcal{R}_{\tilde{X},4} &= -(-b_1 + b_3 + b_5 - b_6)^2 + (-a_1 + a_3 + a_5 - a_6)(-c_1 + c_3 + c_5 - c_6)\,,\\
\mathcal{R}_{\tilde{X},5} &= -(-b_1 + b_2 - b_5 + b_6)^2 + (-a_1 + a_2 - a_5 + a_6)(-c_1 + c_2 - c_5 + c_6)\,,\\
\mathcal{R}_{\tilde{X},6} &= -(-b_1 + b_3 + (b_5 - b_6)\zeta^2)^2 + (-a_1 + a_3 + (a_5 - a_6)\zeta^2)(-c_1 + c_3 + (c_5 - c_6)\zeta^2)\,,\\
\mathcal{R}_{\tilde{X},7} &= -(-b_1 + b_2 - (b_5 - b_6)\zeta^2)^2 + (-a_1 + a_2 - (a_5 - a_6)\zeta^2)(-c_1 + c_2 - (c_5 - c_6)\zeta^2)\,,\\
\mathcal{R}_{\tilde{X},8} &= -(-b_1 + b_2 - (b_5 - b_6)\zeta)^2 + (-a_1 + a_2 - (a_5 - a_6)\zeta)(-c_1 + c_2 - (c_5 - c_6)\zeta)\,,\\
\mathcal{R}_{\tilde{X},9} &= -(-b_1 + b_3 + (b_5 - b_6)\zeta)^2 + (-a_1 + a_3 + (a_5 - a_6)\zeta)(-c_1 + c_3 + (c_5 - c_6)\zeta)\,.
\end{aligned}$$

where $\zeta = e^{2\pi i/3}$.

Introducing a non-zero constant coefficient $A_{\tilde{X},i}$ for every curve, then each Pfaffian is really

$$\mathrm{Pfaff}_{\tilde{X}}(\bar{\partial}_{\tilde{V}_{C_i}(-1)}) = A_{\tilde{X},i}\mathcal{R}_{\tilde{X},i}$$

for $i = 1, \ldots, 9$. Note that we include the factor $1/[\det\bar{\partial}_{\mathcal{O}_{C_i}(-1)}]^2$ in $A_{\tilde{X},i}$.

However, the Beasley-Witten theorem obliquely introduces the constraints that

$$\begin{aligned}
A_{\tilde{X},1} &= -A_{\tilde{X},4} - A_{\tilde{X},5}\,, \\
A_{\tilde{X},2} &= e^{i\pi/3}A_{\tilde{X},4} - A_{\tilde{X},7}\,, \\
A_{\tilde{X},3} &= -e^{i\pi/3}A_{\tilde{X},5} + e^{-i\pi/3}(A_{\tilde{X},4} - A_{\tilde{X},7})\,, \\
A_{\tilde{X},6} &= A_{\tilde{X},4} + A_{\tilde{X},5} - A_{\tilde{X},7}\,, \\
A_{\tilde{X},8} &= e^{i\pi/3}A_{\tilde{X},5} - e^{2i\pi/3}A_{\tilde{X},7}\,, \\
A_{\tilde{X},9} &= A_{\tilde{X},4} + e^{-i\pi/3}(A_{\tilde{X},5} - A_{\tilde{X},7})\,.
\end{aligned}$$

Since we have restricted the polynomials f_1, f_2, f_3 to be $\mathbb{Z}_3 \times \mathbb{Z}_3$ invariant, all of these results apply directly to the quotient space X and the quotient vector bundle V. Using the expression for the sum over the nine torsion homology discussed previously, we find that

$$W_X([C]) = e^{iT^1} \sum_{i=1}^{9} \mathrm{Pfaff}_X(\bar{\partial}_{V_{C_i}(-1)})\chi_i$$

where

$$\mathrm{Pfaff}_X(\bar{\partial}_{V_{C_i}(-1)}) = A_{X,i}\mathcal{R}_{X,i}$$

and

$$\mathcal{R}_{X,i} = \mathcal{R}_{\tilde{X},i}\,, \quad A_{X,i} = A_{\tilde{X},i}$$

It follows that

$$W_X([C]) = e^{iT^1} \sum_{i=1}^{9} \chi_i A_{X,i}\mathcal{R}_{X,i}$$

This does not vanish due to the $\mathbb{Z}_3 \times \mathbb{Z}_3$ characters χ_i . For example, choosing

$$\chi_1 = \chi_2 = \chi_3 = 1 \quad \chi_4 = \chi_5 = \chi_6 = e^{2\pi i/3}\,, \quad \chi_7 = \chi_8 = \chi_9 = e^{4\pi i/3}$$

it follows that

$$W_X([C]) = e^{iT^1}\Big(\sum_{i=1}^{3} A_{X,i}\mathcal{R}_{X,i} + e^{2\pi i/3}\sum_{i=4}^{6} A_{X,i}\mathcal{R}_{X,i} + e^{4\pi i/3}\sum_{i=7}^{9} \mathcal{R}_{X,i}\Big) \neq 0$$

This cannot vanish due to all the previous constraints!