# Anomaly-induced effective action and some applications

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**Partially supported by: CNPq, FAPEMIG**

**QFTG'18, Tomsk – August 1, 2018**

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# Some examples of conformal theories

- **General scalar action with $\xi$ term**

$$S_{scal} = \frac{1}{2} \int d^Dx \sqrt{|g|} \, \{ g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \xi R \phi^2 \} \,, \quad \xi = \frac{D-2}{4(D-1)}$$

**is invariant under local conformal transformation,**

$$g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} \, e^{2\sigma}, \quad \phi \to \phi' = \phi \, e^{n_D \sigma}, \quad n_D = \frac{2-D}{2}, \quad \sigma = \sigma(x) \,.$$

- - **Massless spinor**

$$S_{1/2} = \frac{i}{2} \int d^Dx \sqrt{-g} \, \{ \bar{\psi} \gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma^\mu \psi \}$$

**The transformation rules are**

$$\psi \to \psi' = \psi \, e^{n^f_D \sigma} \,, \quad \bar{\psi} \to \bar{\psi}' = \bar{\psi} \, e^{n^f_D \sigma} \,, \quad n^f_D = \frac{1-D}{2}$$

- **The conformal (Weyl) gravity in the dimension $D = 4$.**

$$S_W \,=\, \int d^4x\sqrt{-g}\,C^2\,,$$

**It can be easily generalized to an arbitrary dimension**

$$C^2(D) \,=\, R^2_{\mu\nu\alpha\beta} - \frac{4}{D-2}\,R^2_{\mu\nu} + \frac{1}{(D-1)(D-2)}\,R^2\,.$$

- **Fourth derivative scalar (can be generalized to dimension $D$ )**

$$S_4 \,=\, \int d^4x\sqrt{-g}\;\varphi\,\Delta_4\,\varphi\,,$$

**where** $\quad \Delta_4 = \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}\,R\Box + \frac{1}{3}\,R_{;\mu}\,\nabla^\mu\,.$

**The transformation law is** $\varphi \to \varphi'\,.$

*E.S. Fradkin & A.A. Tseytlin, PLB,NPB - 1982.*
*S.M. Paneitz, MIT preprint - 1983; SIGMA - 2008*

# Quantum (Semiclassical) Theory

**Introduction:** *Birrell & Davies (1980); Buchbinder, Odintsov & I.Sh. (1992).*

**The most remarkable thing at the quantum level is that the classical conformal invariance is broken (trace anomaly).**

**“Recent” review:** *I.Sh. CQG (2008), gr-qc/0801.0216.*

**Renormalizable theory of massless conformal fields in curved space can be constructed with conformal vacuum terms. At least, only these terms are renormalized at the one-loop level.**

**In a conformal theory at 1-loop level in any even dimension, vacuum divergences satisfy conformal Noether identity,**

$$\frac{1}{\sqrt{|g|}}\, g_{\mu\nu}\, \frac{\delta \Gamma^{(1)}_{div}}{\delta g_{\mu\nu}} \,=\, 0\,.$$

One can distinguish two types of local conformal symmetry in the vacuum sector.

- **C-type local conformal symmetry:**

$$g_{\mu\nu} = \bar{g}_{\mu\nu}\, e^{2\sigma(x)} \quad \Longrightarrow \quad S_C(g_{\mu\nu}) = S_C(\bar{g}_{\mu\nu})\,. \qquad (C)$$

This is a very strong condition.

For $D$-dimensional space this implies cancelation of $1, 2, \dots, D$ powers of $\sigma$ and its derivatives. Example: $\int C^2$-term in $4D$.

- **N-type local conformal symmetry:**

$$S_N(g_{\mu\nu}) \neq S_N(\bar{g}_{\mu\nu})\,, \qquad \textbf{but} \qquad \frac{1}{\sqrt{|g|}}\, g_{\mu\nu}\, \frac{\delta S_N}{\delta g_{\mu\nu}} = 0\,. \qquad (N)$$

Compared to the C-type, here only linear terms should cancel.

Examples: topological and surface terms.

**Classification of terms in anomaly is based on this separation: conformal terms vs topological term vs surface terms.**

*S. Deser, M.J. Duff and C.J. Isham, NPB (1976).*

*S. Deser, and A. Schwimmer, PLB (1993), hep-th/9302047.*

*S. Deser, PLB (2000), hep-th/9911129.*

**There is universality of signs of the coefficients of Weyl invariant (C-type) and topological terms.**

**The related property of the renormalization group flows probably holds beyond the one-loop level, at least for some QFT models.**

**This opened a new area which is known as *c*- and *a*-theorems,**

*Z. Komargodski, A. Schwimmer, JHEP (2011), arXiv:1107.3987.*

*M.A. Luty, J. Polchinski, R. Rattazzi, JHEP (2013), arXiv:1204.5221.*

*Z. Komargodski, JHEP (2012), arXiv:1112.4538.*

# Particular dimensions

2D. **There are no C-type invariants and only one N-type invariant,**

$$S_2(g_{\mu\nu}) = \int d^2x\sqrt{|g|}\,R \quad \Longrightarrow \quad \langle T^{\mu}_{\mu}\rangle = aR\,,$$

**Integration of anomaly is simple and yields the Polyakov action**

$$\Gamma_{ind}(g_{\mu\nu}) = \frac{a}{4}\int d^2x\sqrt{|g|}\,R\,\frac{1}{\Box}\,R\,,$$

*A.M. Polyakov, PLB (1981).*

4D. **There is one C-type invariant, and two N-type invariants,**

$$\langle T^{\mu}_{\mu}\rangle = -\omega C^2 - bE - c\Box R\,,$$

**where**

$$\begin{pmatrix}\omega\\ b\\ c\end{pmatrix} = \frac{1}{360(4\pi)^2}\begin{pmatrix}+3N_0 + 18N_{1/2} + 36N_1\\ -N_0 - 11N_{1/2} - 62N_1\\ +2N_0 + 12N_{1/2} - 36N_1\end{pmatrix}.$$

**In 4D we can obtain the non-local covariant solution for the anomaly-induced effective action of vacuum.**

**First one has to establish the relations**

$$\sqrt{-g}C^2 = \sqrt{-\bar{g}}\bar{C}^2\,, \qquad \sqrt{-\bar{g}}\,\bar{\Delta}_4 = \sqrt{-g}\,\Delta_4\,,$$

$$\sqrt{-g}\Big(E - \frac{2}{3}\Box R\Big) \;=\; \sqrt{-\bar{g}}\Big(\bar{E} - \frac{2}{3}\bar{\Box}\bar{R} + 4\bar{\Delta}_4\sigma\Big) \quad \textbf{!!}$$

**and also introduce the Green function**

$$\sqrt{-g}\Delta_4\,G(x,y) = \delta(x,y)\,.$$

**Using these formulas we find, for a functional** $A(g_{\mu\nu}) = A(\bar{g}_{\mu\nu})$,

$$\frac{\delta}{\delta\sigma}\int_x A\;\Big(E - \frac{2}{3}\Box R\Big)\Big| \;=\; 4\sqrt{-g}\Delta_4\,A\,,$$

**where** $$\int_x \;=\; \int d^4x\,\sqrt{-g(x)}\,, \qquad \Big| \;=\; \Big|_{\bar{g}_{\mu\nu}\to g_{\mu\nu}}\,.$$

**As a consequence, we obtain**

$$\frac{\delta}{\delta\sigma(y)}\iint\limits_{x\,z}\frac{1}{4}C^2(x)\,G(x,z)\,\Big(E-\frac{2}{3}\Box R\Big)_z\Big|$$

$$=\int d^4x\sqrt{-\bar{g}(x)}\,\bar{\Delta}_4(x)\,\bar{G}(x,y)\,\bar{C}^2(x)\Big|\;=\;\sqrt{-g(y)}\,C^2(y)\,.$$

**Hence, the part of $\Gamma_{ind}$ which is responsible for $T_\omega=-\omega C^2$, is**

$$\Gamma_\omega=\frac{\omega}{4}\iint\limits_{x\,y}C^2(x)\,G(x,y)\,\Big(E-\frac{2}{3}\Box R\Big)_y=\frac{\omega}{4}\iint\limits_{x\,y}C^2(x)\,G(x,y)\,\tilde{E}_*(y)\,.$$

**Similarly, one can check that the variation $T_b=-b\,\tilde{E}_*$ is produced by the term**

$$\Gamma_b\;=\;\frac{b}{8}\iint\limits_{x\,y}\tilde{E}_*(x)\,G(x,y)\,\tilde{E}_*(y).$$

**Finally, we can use simple relation**

$$g_{\mu\nu}\,\frac{\delta}{\delta g_{\mu\nu}}\,\int_x R^2(x)\,=\,-6\,\sqrt{-g}\,\Box R\,.$$

**to establish the remaining local constituent of $\Gamma_{ind}$**

$$\Gamma_c=-\frac{3c+2b}{36(4\pi)^2}\,\int_x R^2(x)\,.$$

**The general covariant solution for $\Gamma_{ind}$ is the sum,**

$$\Gamma_{ind}\,=S_c[g_{\mu\nu}]\,-\,\frac{3c+2b}{36(4\pi)^2}\,\int_x R^2(x)$$

$$+\,\frac{\omega}{4}\,\int_x\int_y C^2(x)\,G(x,y)\,\tilde{E}_*(y)\,+\,\frac{b}{8}\,\int_x\int_y \tilde{E}_*(x)\,G(x,y)\,\tilde{E}_*\,.$$

**One can rewrite this expression using auxiliary scalars.**

**The nonlocal terms can be recast into symmetric form**

$$E_*(x)\, G(x,y) \left(\frac{\omega}{4}C^2 - \frac{b}{8}\tilde{E}_*\right)_y$$

$$= \frac{b}{8}\iint\limits_{x\,y} \left(\tilde{E}_* - \frac{\omega}{b}C^2\right)_x G(x,y)\left(\tilde{E}_* - \frac{\omega}{b}C^2\right)_y - \frac{\omega^2}{8b}\iint\limits_{x\,y} C_x^2\, G(x,y)\, C_y^2\,.$$

**Thus we arrive at the local covariant expression for EA**

$$\Gamma_{ind} = S_c[g_{\mu\nu}] - \frac{3c+2b}{36(4\pi)^2}\int\limits_x R^2(x) + \int\limits_x \Big\{ \frac{1}{2}\,\varphi\Delta_4\varphi - \frac{1}{2}\,\psi\Delta_4\psi$$

$$+ \frac{\omega}{8\pi\sqrt{-b}}\,\psi\, C^2 + \varphi\left[\frac{\sqrt{-b}}{8\pi}\left(E - \frac{2}{3}\Box R\right) - \frac{\omega}{8\pi\sqrt{-b}}\, C^2\right]\Big\}\,.$$

**This is the best form for $\Gamma_{ind}$.**
*I.Sh. and A.Jacksenaev, Phys. Lett. B (1994).*

**Similar expression has been independently introduced by**
*P. Mazur & E. Mottola, 1997-1998-2001.*

# Recent generalization

**Quantum effects of chiral fermion field produce an imaginary contribution which violates parity,**

$$\langle T^\mu_\mu \rangle \;=\; -\,\omega_1 C^2 - bE_4 - c\Box R - \epsilon P_4\,,$$

**where the Pontryagin density term appears,**

$$P_4 \;=\; \frac{1}{2}\,\varepsilon^{\mu\nu\alpha\beta}\,R_{\mu\nu\rho\sigma}\,R_{\alpha\beta}{}^{\rho\sigma}\,, \qquad \epsilon = \frac{i}{48\cdot 16\pi^2}\,.$$

*L. Bonora, S. Giaccari, B. de Souza, JHEP (2014), arXiv:1403.2606.*

*L. Bonora, M. Cvitan, P. Dominis Prester, A. Duarte Pereira,*
*S. Giaccari, T. Stemberga, Eur.Phys.J.C (2017) arXiv:1703.10473.*

**It is an easy exercise to derive the corresponding anomaly-induced effective action.**

*S. Mauro, I.Sh., PLB (2015), arXiv:1412.5002.*

**First, one can prove the conformal symmetry of this term**

$$P_4 = \frac{1}{2}\,\varepsilon^{\mu\nu\alpha\beta}\,C_{\mu\nu\rho\sigma}\,C_{\alpha\beta}{}^{\rho\sigma}\,.$$

**After that we immediately arrive at**

$$\Gamma_{ind} \;=\; S_c[g_{\mu\nu}] - \frac{3c+2b}{36(4\pi)^2}\int_x R^2 + \int_x \Big\{\frac{1}{2}\,\varphi\Delta_4\varphi - \frac{1}{2}\,\psi\Delta_4\psi$$

$$+\;\frac{\sqrt{-b}}{8\pi}\big(E - \frac{2}{3}\,\Box R\big)\varphi \;+\; \frac{1}{8\pi\sqrt{-b}}\,\big(\omega C^2 + \epsilon P_4\big)\,\big(\psi - \varphi\big)\Big\}.$$

**It is natural to change variables,**

$$\chi = \frac{\psi - \varphi}{\sqrt{2}}\,, \qquad \xi = \frac{\psi + \varphi}{\sqrt{2}}\,,$$

**Then the total gravitational action becomes**

$$\Gamma_{grav} \;=\; S_{EH} + S_{HD} + S_c[g_{\mu\nu}] \;+\; \int_x \Big\{\xi\Delta_4\chi \;+\; k_1\big(E - \frac{2}{3}\,\Box R\big)\big(\xi - \chi\big)$$

$$+\,k_2\,\chi\,\big(\omega C^2 + \epsilon P_4\big) \;+\; k_3 R^2\Big\}\,.$$

**The coefficients are, as before,**

$$k_1 = \frac{1}{8\pi}\sqrt{-\frac{b}{2}}\,, \qquad k_2 = \frac{1}{8\pi\sqrt{-2b}}\,, \qquad k_3 = -\frac{2b+3c}{36\,(4\pi)^2}\,.$$

**The action**

$$\Gamma_{grav} = S_{EH} + S_{HD} + S_c[g_{\mu\nu}] + \int_x \Big\{ \underline{\xi\Delta_4\chi} + k_1\big(E - \frac{2}{3}\Box R\big)(\xi - \chi)$$

$$+ k_2\,\chi\,\left(\omega C^2 + \underline{\epsilon P_4}\right) + k_3 R^2 \Big\}\,.$$

**is a special case of the Chern-Simons modified general relativity,**

*R. Jackiw and S.Y. Pi, Phys. Rev. D 68 (2003), gr-qc/0308071.*

*A. Lue, L. Wang, M. Kamionkowski, Phys.Rev.Lett. 83 (1999) 1506.*

*S. Alexander, N. Yunes, Phys.Rept. 480 (2009) 1.*

**with a particular form of the kinetic term.**

# General perspective: a few mysteries remains

- **Why the beta-functions for conformal and topological terms, which come from fields of all spins, have the same sign?**

- **And why there exist the "main formula"**

$$\sqrt{-g}\Big(E - \frac{2}{3}\Box R\Big) \,=\, \sqrt{-\bar{g}}\Big(\bar{E} - \frac{2}{3}\bar{\Box}\bar{R} + 4\bar{\Delta}_4\sigma\Big) \quad \textbf{??}$$

**And why the "magic" $2/3$ coefficient? It there some general mathematical rule behind this occurrence?**

- **Since our knowledge is restricted, we need further examples.**
- **But before that let us consider some general features.**

$\forall D$. **Several C-type invariants, and two kinds of N-type invariants,**

$$T \,=\, \langle T^\mu_\mu \rangle \,=\, c_r\, W^r_D \,+\, a\, E_D \,+\, \Xi_D\,,$$

**with the sum over** $r$. **Here** $W^r_D$ **are conformal invariant terms and**

$$E_D \,=\, D^{-1}\, \varepsilon^{\rho_1\cdots\rho_D}\, \varepsilon^{\sigma_1\ldots\sigma_D}\, R_{\rho_1\sigma_1\rho_2\sigma_2}\cdots R_{\rho_{D-1}\sigma_{D-1}\rho_D\sigma_D}.$$

**Example:** $\underline{6D}$. **There are three C-type terms,**

$$W_6^1 = C_{\mu\nu\rho\sigma}\, C^{\mu\alpha\beta\nu}\, C_{\alpha\,\cdot\,\cdot\,\beta}^{\;\;\rho\sigma}\,,$$

$$W_6^2 = C_{\mu\nu\rho\sigma}\, C^{\rho\sigma\alpha\beta}\, C_{\alpha\beta\,\cdot\,\cdot}^{\;\;\;\;\mu\nu}\,,$$

$$W_6^3 = C_{\mu\rho\sigma\lambda}\Big(\delta_\nu^\mu + 4R_\nu^\mu - \frac{6}{5}\,R\,\delta_\nu^\mu\Big)\,C^{\nu\rho\sigma\lambda} \,+\, \nabla_\mu J^\mu\,,$$

**where** $$J_\mu \,=\, \big(4R_\mu^{\;\cdot\lambda\rho\sigma}\nabla^\nu + 3R^{\nu\lambda\rho\sigma}\nabla_\mu\big)R_{\nu\lambda\rho\sigma}$$

$$+\,\Big(\frac{1}{2}R\nabla_\mu - R_\mu^\nu\nabla_\nu\Big)R + R^{\nu\lambda}\big(\nabla_\nu R_{\lambda\mu} - 5\nabla_\mu R_{\nu\lambda}\big).$$

**This is a more complicated case. There are also several N-type terms, including one topological and others total derivatives.**

*F. Bastianelli, S. Frolov, A. Tseytlin, JHEP (2000), hep-th/0001041.*

**For a general dimension,** $\Xi_D$ **is a linear combination of surface terms,** $\Xi_D = \sum \gamma_k \chi_k$. **The numerical coefficients** $a,\, c,\, \gamma_k$ **depend on the number of massless conformal fields of different spins.**

**Our purpose is to find the anomaly-induced EA $\Gamma_{ind}$ , such that**

$$-\frac{2}{\sqrt{-g}}\,g_{\mu\nu}\,\frac{\delta\Gamma_{ind}}{\delta g_{\mu\nu}} \,=\, T\,.$$

**Integration of anomaly requires modified topological invariant**

$$\tilde{E}_D = E_D + \sum \alpha_k \chi_k\,, \qquad (H)$$

**where the values of $\alpha_k$ are chosen to provide the special conformal property of the new topological term.**

**Under the local conformal transformation**

$$g_{\mu\nu} = \bar{g}_{\mu\nu}\,e^{2\sigma(x)}$$

**there should be**

$$\sqrt{-g}\tilde{E}_D \,=\, \sqrt{-\bar{g}}\big(\bar{\tilde{E}}_D + \kappa\bar{\Delta}_D\sigma\big)\,,$$

**where $\kappa$ is a constant and $\Delta_D = \Box^{D/2} + \dots$ is a conformal operator acting on a conformally inert scalar.**

**The existence of such a modified topological term** (H) **for general *D* is no more than an important conjecture.**

**In order to integrate anomaly one has to find local metric dependent Lagrangians $\mathcal{L}_i$ providing that, with some coefficients $c_{ik}$ there would be an identity (why should it be?)**

$$-\frac{2}{\sqrt{-g}}\,g_{\mu\nu}\,\frac{\delta}{\delta g_{\mu\nu}}\,\sum_i c_{ik}\int_x \mathcal{L}_i \,=\, \chi_k\,.$$

**After that the problem is reduced to integrating the first two terms in anomaly, by using Green function**

$$\sqrt{-g}\,\Delta_D^x\,G(x,x')\,=\,\delta^D(x,x')\,,\qquad G\,=\,\bar{G}\,.$$

**The non-local solution for the anomaly-induced EA is**

$$\Gamma_{ind}\,=\,S_c+\iint_{x\,y}\Big\{\frac{1}{4}\,c_r\,W_D^r\,+\,\frac{a}{8}\,\tilde{E}_D(x)\Big\}\,G(x,y)\,\tilde{E}_D(y)$$

$$+\sum_k\big(\gamma_k-\alpha_k\big)\sum_i c_{ik}\int_x \mathcal{L}_i\,.$$

**The modification of the coefficients $\gamma_k$ takes place because part of the surface terms were absorbed into $\tilde{E}_D$.**

**Writing the non-local part of the induced effective action in the symmetric form, one can construct a local covariant presentation with the use of two auxiliary fields $\psi$, $\varphi$,**

$$\bar{\Gamma} = S_c + \sum_k (\gamma_k - \alpha_k) \sum_i c_{ik} \int_x \mathcal{L}_i$$

$$+ \frac{1}{2} \int_x \Big\{ \varphi \Delta_D \varphi - \psi \Delta_D \psi + \sqrt{-a}\, \varphi \, \tilde{E}_D + \frac{1}{\sqrt{-a}} (\psi - \varphi)\, c_r \, W_D^r(x) \Big\} .$$

**Here we assumed $a < 0$, as in the $D = 4$ case. For the opposite sign one can modify $\tilde{E}_D \to -\tilde{E}_D$.**

**We obtained explicit representations of the EA which corresponds to the general dimension-independent structure of trace anomaly established by Deser and Schwimmer.**

**It is remarkable that the leading UV part of the vacuum effective action in an arbitrary even dimension can be given in such a simple and general form.**

# On the big mystery on conformal anomaly: 6*D*

**Recently the explicit solution for** 6*D* **has been obtained, after about four years of heavy work.**

*Fabricio Ferreira, I.Sh, PLB (2017), arXiv:1702.06892.*

**See also**

*F. Ferreira, I.Sh, P. Teixeira, EPhJ.Plus* **131** *(2016) , arXiv:1507.03620.*

**The candidate terms to the total derivatives in $\langle T \rangle$ can be reduced to the form**

$$\chi_1 = \Box^2 R\,, \quad \chi_2 = \Box R^2_{\mu\nu\alpha\beta}\,, \quad \chi_3 = \Box R^2_{\mu\nu}\,, \quad \chi_4 = \Box R^2\,,$$

$$\chi_5 = \nabla_\mu \nabla_\nu \big( R^{\mu}{}_{\lambda\alpha\beta} R^{\nu\lambda\alpha\beta} \big); \qquad \chi_6 = \nabla_\mu \nabla_\nu \big( R_{\alpha\beta} R^{\mu\alpha\nu\beta} \big)$$

$$\chi_7 = \nabla_\mu \nabla_\nu \big( R^{\mu}_{\alpha} R^{\nu\alpha} \big); \qquad \chi_8 = \nabla_\mu \nabla_\nu \big( R R^{\mu\nu} \big)\,.$$

**The result for $D=6$ is**

$$\tilde{E}_D = E_D + \sum_k \alpha_k \chi_k \,, \qquad (H)$$

**with**

$$\alpha_1 = \frac{3}{5}\,, \quad \alpha_2 = -\frac{9}{10} - \frac{5}{4}\xi_1 + \frac{3}{8}\xi_2\,, \quad \alpha_3 = \xi_1\,, \quad \alpha_4 = 0\,,$$

$$\alpha_5 = \frac{84}{5} + 3\xi_1 + \frac{11}{2}\xi_2\,, \quad \alpha_6 = -\frac{36}{5} - 2\xi_1 - 5\xi_2\,,$$

$$\alpha_7 = -\frac{18}{5} - \xi_1 - \frac{7}{2}\xi_2\,, \quad \alpha_8 = \xi_2\,.$$

**Here $\xi_1,\ \xi_2$ are arbitrary parameters.**

**With this choice, all the non-linear in $\sigma$ terms cancel, and the remaining linear term corresponds to conformal operator $\Delta_6$.**

$$\sqrt{-g}\tilde{E}_D \;=\; \sqrt{-\bar{g}}\big(\bar{\tilde{E}}_D + 6\bar{\Delta}_D\sigma\big)\,.$$

**The conformal operator is (coincides with** *K. Hamada, Prog. Theor. Phys. (2001), hep-th/0012053* **)**

$$\Delta_6 = \Box^3 + 4R^{\mu\nu}\nabla_\mu\nabla_\nu\Box - R\Box^2 + 4(\nabla^\alpha R^{\mu\nu})\nabla_\alpha\nabla_\mu\nabla_\nu + V^{\mu\nu}\nabla_\mu\nabla_\nu + N^\lambda\nabla_\lambda\,,$$

**where** $V^{\mu\nu} = \Big(\frac{78}{5} + \frac{3\xi_2 + 2\xi_1}{2}\Big)\Big(R^{\mu\alpha}R^\nu_\alpha - \frac{1}{3}RR^{\mu\nu}\Big)$

$$+\Big[\Big(1+\frac{\xi_2}{6}\Big)R^2 - \Big(\frac{29}{5} + \frac{17\xi_2 - 2\xi_1}{12}\Big)R^2_{\rho\sigma} + \Big(\frac{16}{5} + \frac{7\xi_2 - 2\xi_1}{6}\Big)R^2_{\rho\sigma\alpha\beta} - \frac{3}{5}\Box R\Big]\, g^{\mu\nu}$$

$$+\Big(\frac{64}{5} + \frac{4\xi_1}{3} + 2\xi_2\Big)\Big(R_{\alpha\beta}R^{\mu\alpha\nu\beta} - R^\mu{}_{\alpha\beta\gamma}R^{\nu\alpha\beta\gamma}\Big)$$

**and**

$$N^\lambda = \frac{2}{5}(\nabla^\lambda\Box R) + \frac{8}{3}(\xi_1 - \xi_2)R_{\alpha\beta\rho\sigma}(\nabla^\rho R^{\alpha\beta\sigma\lambda}) - \Big(\frac{3}{5} + \frac{\xi_1}{6} - \frac{\xi_2}{12}\Big)R(\nabla^\lambda R)$$

$$+\Big(\frac{14}{5} - \frac{\xi_1}{3} - \frac{\xi_2}{2}\Big)R_{\rho\sigma}(\nabla^\rho R^{\sigma\lambda}) + \Big(\frac{6}{5} + \frac{5\xi_1}{3} - \frac{5\xi_2}{6}\Big)R_{\rho\sigma}(\nabla^\lambda R^{\rho\sigma})$$

$$+\Big(\frac{13}{5} + \frac{\xi_1}{6} + \frac{\xi_2}{4}\Big)R^{\rho\lambda}(\nabla_\rho R) + \Big(\frac{64}{5} + \frac{4\xi_1}{3} + 2\xi_2\Big)R^{\rho\sigma\alpha\lambda}(\nabla_\rho R_{\sigma\alpha})\,.$$

- **Cosmological Model based on the action**

$$S_{total} \,=\, -\,\frac{M_P^2}{16\pi}\,\int d^4x\sqrt{-g}\,(R+2\Lambda)+S_{matter}+S_{vac}+\bar{\Gamma}_{ind}\,.$$

**Equation of motion for** $a(t),\quad dt=a\,(\eta)\,d\eta,\quad k=0$

$$\frac{\ddddot{a}}{a}+\frac{3\dot{a}\dddot{a}}{a^2}+\frac{\ddot{a}^2}{a^2}-\left(5+\frac{4b}{c}\right)\frac{\ddot{a}\dot{a}^2}{a^3}-\frac{M_P^2}{8\pi c}\left(\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}-\frac{2\Lambda}{3}\right)\,=\,0\,,$$

**Particular solutions (Starobinsky, PLB-1980)**

$$a(t)\,=\,a_0e^{Ht}\,,$$

**where Hubble parameter** $H=\dot{a}/a$ **is**

$$H^2\,=\,-\,\frac{M_P^2}{32\pi b}\left(1\pm\sqrt{1+\frac{64\pi b}{3}\,\frac{\Lambda}{M_P^2}}\,\right).$$

**Case of** $\Lambda\neq 0$ **in** *A. Pelinson, I.Sh., F. Takakura, NPB-2003.*

**For $0 < \Lambda \ll M_P^2$ there are two solutions:**

$$H \approx \sqrt{\Lambda/3}\,; \qquad\qquad (IR)$$

$$H \approx \sqrt{-\frac{M_P^2}{16\pi b} - \frac{\Lambda}{3}} \approx \frac{M_P}{\sqrt{-16\pi b}}\,. \qquad\qquad (UV)$$

**Perturbations of the conformal factor**

$$\sigma(t) \to \sigma(t) + y(t).$$

**The criterion for a stable $(UV)$ inflation is**

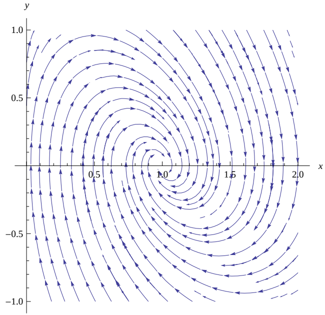
$$c > 0 \quad \Longleftrightarrow \quad N_1 < \frac{1}{3}N_{1/2} + \frac{1}{18}N_0\,,$$

**in agreement with Starobinsky (1980).**

**The original Starobinsky model is based on the unstable case and involves special choice of initial data. This situation can be improved further by using the stable version and an appropriate transition scheme.**

**In the unstable phase there are very different solutions, some of them violent (hyperinflation). How can we know that the transition from stable to unstable phase really happens?**

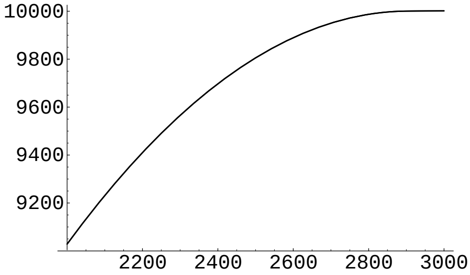
*A. Pelinson et al, NPB(PS) (2003):* **Phase portrait of a stable case:**




**Starobinsky (1980)** $x = \left(\frac{H}{H_0}\right)^{\frac{3}{2}}, \quad y = \frac{\dot{H}}{2\sqrt{H_0^3 H}}, \quad dt = \frac{dx}{3H_0\, x^{2/3}\, y}.$

## Anomaly-induced inflation slows down if taking masses of quantum fields into account.

*Sh., Solà, Phys.Lett. 530B (2002);*
*Pelinson, Sh. & Takakura, Nucl.Ph. 648B (2003).*

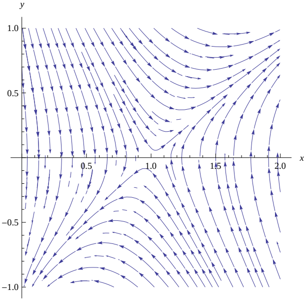


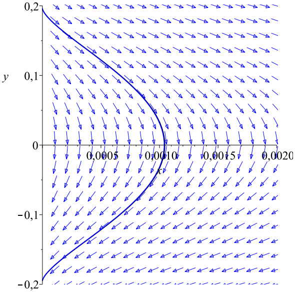

$$\sigma(t) = \ln a(t) \approx H_0 t - \frac{H_0^2}{4} \tilde{f} \, t^2 , \qquad H_0 \propto M_P$$

**The total amount of *e*-folds may be as large as** $10^{32}$**, but only** 65 **last ones, where** $H \propto M_*$ **(SUSY breaking scale) are relevant.**

## Phenomenological approach to the transition from stable to unstable inflation.

*A. Pelinson, Tibério de Paula Netto, I.Sh., A. Starobinsky, EJPhC (2016), arXive:1509.08882.*




**The plot really ends up at the classical radiation-dominated solution.**

**This result gives us a chance to have a consistent inflation based on QFT results.**

# Conclusions

- **Integrating conformal anomaly is an efficient, economic and simple way to derive the the non-local part of the effective action of vacuum. And it is mostly reliable.**

- **There are many generalizations of the original method. One can even obtain the result for the light massive fields, if treating masses as small perturbations.**

- **The main application of this method is the Starobinsky model, which is capable to provide a direct link between QFT methods and the background of cosmology.**

- **From the Math. side there is a general conjecture about the conformal feature of topological terms, which would be great to prove. At the moment we have positive answers in $D = 2, 4, 6$.**