Extended supersymmetric multidimensional mechanics and Calogero-like models

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Plan

- Motivation
- Euler-Calogero-Moser model
  - Bosonic case
  - Model with $\mathcal{N}=2$ supersymmetry
  - Superconformal invariance
  - Model with even number of supersymmetries
  - $Osp(2M|2)$ superconformal symmetry of the model
- $\mathcal{N}=2$ supersymmetric Euler-Calogero-Moser model in superspace
  - Superfield constraints and action
  - Component action
- Concluding remarks

I. Motivation

It is well known that integrable systems of the spin-Calogero type can be obtained from matrix models. Usually, when constructing the action of matrix models (symmetric, Hermitian, unitary, etc.), the corresponding matrices are chosen as products that can be schematically written as

$$\begin{array}{ll} M = U^{\dagger}\, D\, U & \text{for Hermitian matrix model,} \\ M = O^{T}\, D\, O & \text{for symmetric matrix model,} \end{array} \qquad (1)$$

where U, O and D are unitary, ortogonal and diagonal matrices, respectively. Since the actions of these models are provided by the following integral

$$S_M = \int dt\, \mathrm{Tr}\big(\dot{M}\dot{M}\big), \qquad (2)$$

the contribution to the kinetic term is connected only with components of the diagonal matrix D. While the components of the other matrices provide the spin part of the model.

When constructing supersymmetric extensions of matrix models, the original matrices are generalized to supermatrices, which are now functions on the superspace.
For example, in the $\mathcal{N}=2$ supersymmetric case each component of matrices is some superfield defined by

$$x = x(t, \theta, \bar{\theta}), \tag{3}$$

the integration measure is changed to

$$\int dt \quad \rightarrow \quad \int dt d\theta d\bar{\theta} \tag{4}$$

and the Lagrangian acquires the form

$$\mathcal{L}^{\mathcal{N}=2} = \mathrm{Tr}\big(DM\bar{D}M\big). \tag{5}$$

After integrating over Grassmann variables, in addition to the standard kinetic terms of boson fields, a large set of fermion fields with time derivatives, associated with the off-diagonal part of the matrix M, appears in the component action.

Thus, a supersymmetric matrix model is always accompanied by additional fermions that can represent the lower components of the corresponding fermionic superfields.

The Lagrangian approach is not very useful in the extended supersymmetry because of the presence of huge set of Grassmann variables. Therefore, in constructing a supersymmetric extension of the Euler-Calogero-Moser model, we will follow methods of Hamiltonian mechanics.

Main result

We propose an extended, with even number of supersummetries, *n*-particles Euler-Calogero-Moser model by using the Hamiltonian approach.
Then we construct a set of conserved currents and demonstrate that they from the $osp(2M|2)$ superconformal algebra, which is an algebra of the symmetry group of the $\mathcal{N} = 2M$ supersymmetric Euler-Calogero-Moser model.

We also give a superfield description for the case of $\mathcal{N} = 2$ supersymmetric extension of the *n*-particles Euler-Calogero-Moser model and, finally, find the corresponding component action.

II. Euler-Calogero-Moser model
Bosonic case

The spin generalization of the *n*-particles Calogero-Moser model, which is also known as the Euler-Calogero-Moser model, is described by the following Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{2}\sum_{i\neq j}^{n} \frac{\ell_{ij}^2}{(x_i - x_j)^2}\,. \tag{6}$$

It depends on coordinates $x_i(t)$ and momenta $p_i(t)$ of each particle as well as on the internal degrees of freedom encoded in the angular momentum $\ell_{ij} = -\ell_{ji}$.
The coordinates and momenta satisfy the standard Poisson brackets

$$\{x_i, p_j\} = \delta_{ij}, \tag{7}$$

while the the Poisson brackets of the angular momenta form the $so(n)$ algebra

$$\{\ell_{ij}, \ell_{km}\} = \frac{1}{2}\left(\delta_{ik}\ell_{jm} + \delta_{jm}\ell_{ik} - \delta_{jk}\ell_{im} - \delta_{im}\ell_{jk}\right). \tag{8}$$

A model with the Hamiltonian (6) possesses conformal invariance. If we define the conserved currents of dilatation D and conformal boosts K as

$$D = -\frac{1}{2}\sum_{i=1}^{n} x_i p_i + tH, \qquad K = \frac{1}{2}\sum_{i=1}^{n} x_i^2 - t\sum_{i=1}^{n} x_i p_i + t^2 H, \tag{9}$$

then it is easy to demonstrate that they form together with the Hamiltonian H (6) the one-dimensional conformal algebra $so(1,2)$:

$$\{H, K\} = 2D, \quad \{H, D\} = H, \quad \{K, D\} = -K. \tag{10}$$

Model with $\mathcal{N}=2$ supersymmetry

The $\mathcal{N}=2$ supersymmetric extension of the n-particles Euler-Calogero-Moser model is described by two supercharges $Q,\overline{Q}$ and Hamiltonian H, whose bosonic limit is (6), and which form $\mathcal{N}=2$ super Poincaré algebra

$$\{Q,\overline{Q}\}=-2\mathrm{i}H, \quad \{Q,Q\}=\{\overline{Q},\overline{Q}\}=0. \tag{11}$$

In order to construct this extended model, it is necessary to introduce a certain set of fermionic components. First of all, we have to add $2n$ fermion fields $\psi_i(t),\bar{\psi}_i(t)$, satisfied the Poisson brackets

$$\{\psi_i,\bar{\psi}_j\}=-\mathrm{i}\delta_{ij}. \tag{12}$$

These fermions can be combined with the bosonic coordinates $x_i(t)$ into the $\mathcal{N}=2$ supermultiplets. Nevertheless, this set of fermions is not enough to realize the superchargers $Q,\overline{Q}$ in such a way that their anticommutator could lead to the correct bosonic limit of the potential part in Hamiltonian (6).

Indeed, to produce the potential term $\sum_{i>j}^{n}\frac{\ell_{ij}^{2}}{(x_i-x_j)^2}$, the supercharges should, in particular, contain linear in symmetric spinor fields $\rho_{ij}(t),\bar\rho_{ji}(t)$ terms of the following type

$$Q\sim\frac{\rho_{ij}\ell_{ij}}{x_i-x_j},\qquad \overline{Q}\sim\frac{\bar\rho_{ij}\ell_{ij}}{x_i-x_j}. \tag{13}$$

The conjecture that these symmetric spinors are constructed as a sum of the fermionic components as

$$\rho_{ij}=\psi_i+\psi_j,\quad \bar\rho_{ij}=\bar\psi_i+\bar\psi_j \tag{14}$$

leads to inconsistency of the possible structure of supercharges with the basic relation (11) of the $\mathcal{N}=2$ super Poincaré algebra. Therefore, we will treat the spinors $\rho_{ij}(t),\bar\rho_{ji}(t)$ as new fields in addition to $\psi_i(t),\bar\psi_i(t)$. The spinors $\rho_{ij}(t)$, $\bar\rho_{ji}(t)$ satisfy the condition $\rho_{ii}=\bar\rho_{ii}=0$ for each indices i and obey the following Poisson brackets

$$\{\rho_{ij},\bar\rho_{km}\}=-\frac{\mathrm{i}}{2}\left(\delta_{ik}\delta_{jm}+\delta_{im}\delta_{jk}\right). \tag{15}$$

Thus, a complete counting of the fermionic degrees of freedom leads to the result that there are $n(n+1)$ fermionic fields in the model: $(\psi_i,\bar\psi_i)=2n,(\rho_{ij},\bar\rho_{ij})=n(n-1)$.

The next important ingredient in the construction of supercharges is the composite object $\Pi_{ij}=-\Pi_{ji}$

$$\Pi_{ij}=-\mathrm{i}\Big[\left(\psi_i-\psi_j\right)\bar\rho_{ij}+\left(\bar\psi_i-\bar\psi_j\right)\rho_{ij}+\sum_{k=1}^{n}\left(\rho_{ik}\bar\rho_{kj}-\rho_{jk}\bar\rho_{ki}\right)\Big]. \qquad (16)$$

One may check that with respect to the brackets (12), (15) the Π_{ij} form the same $so(n)$ algebra as ℓ_{ij}

$$\left\{\Pi_{ij},\Pi_{km}\right\}=\frac{1}{2}\left(\delta_{ik}\Pi_{jm}+\delta_{jm}\Pi_{ik}-\delta_{jk}\Pi_{im}-\delta_{im}\Pi_{jk}\right). \qquad (17)$$

Now, it is a matter of straightforward calculations to check that the supercharges Q and $\overline{Q}$ given by

$$Q=\sum_{i=1}^{n}p_i\psi_i-\sum_{i\neq j}^{n}\frac{\left(\ell_{ij}+\Pi_{ij}\right)\rho_{ij}}{x_i-x_j}\,, \qquad \overline{Q}=\sum_{i=1}^{n}p_i\bar\psi_i-\sum_{i\neq j}^{n}\frac{\left(\ell_{ij}+\Pi_{ij}\right)\bar\rho_{ij}}{x_i-x_j} \qquad (18)$$

together with the Hamiltonian

$$H=\frac{1}{2}\sum_{i=1}^{n}p_i^2+\frac{1}{2}\sum_{i\neq j}^{n}\frac{\left(\ell_{ij}+\Pi_{ij}\right)^2}{\left(x_i-x_j\right)^2} \qquad (19)$$

form $\mathcal{N}=2$ super Poincaré algebra (11) and describe the $\mathcal{N}=2$ supersymmetric extension of the n-particles Euler-Calogero-Moser model.

Superconformal invariance

As was discussed before, the n-particles Euler-Calogero-Moser model is conformally invariant. Therefore, we expect that its $\mathcal{N}=2$ supersymmetric extension also possesses superconformal symmetry. Indeed, one can show that the set of generators, which includes the supercharges $Q, \overline{Q}$ (18), Hamiltonian H (19) and the following conserved currents

$$K=\frac{1}{2}\sum_{i=1}^{n} x_i^2 - t\sum_{i=1}^{n} x_i p_i + t^2 H, \quad D=-\frac{1}{2}\sum_{i=1}^{n} x_i p_i + tH, \tag{20}$$

$$U=-\frac{1}{2}\sum_{i=1}^{n}\psi_i\bar{\psi}_i - \frac{1}{2}\sum_{i\neq j}^{n}\rho_{ij}\bar{\rho}_{ij}, \quad S=\sum_{i=1}^{n} x_i\psi_i - tQ, \quad \overline{S}=\sum_{i=1}^{n} x_i\bar{\psi}_i - t\overline{Q},$$

form the $osp(2|2)\sim su(1,1|1)$ superconformal algebra

$$\{H,K\}=2D, \quad \{H,D\}=H, \quad \{K,D\}=-K, \tag{21}$$

$$\{U,Q\}=\frac{\mathrm{i}}{2}Q,\ \{U,\overline{Q}\}=-\frac{\mathrm{i}}{2}\overline{Q}, \quad \{U,S\}=\frac{\mathrm{i}}{2}S,\ \{U,\overline{S}\}=-\frac{\mathrm{i}}{2}\overline{S},$$

$$\{D,Q\}=-\frac{1}{2}Q,\ \{D,\overline{Q}\}=-\frac{1}{2}\overline{Q}, \quad \{D,S\}=\frac{1}{2}S,\ \{D,\overline{S}\}=\frac{1}{2}\overline{S},$$

$$\{H,S\}=-Q,\ \{H,\overline{S}\}=-\overline{Q}, \quad \{K,Q\}=S,\ \{K,\overline{Q}\}=\overline{S},$$

$$\{Q,\overline{Q}\}=-2\mathrm{i}H,\ \{S,\overline{S}\}=-2\mathrm{i}K, \quad \{Q,\overline{S}\}=2\mathrm{i}D+2U,\ \{\overline{Q},S\}=2\mathrm{i}D-2U.$$

Model with even number of supersymmetries

The $\mathcal{N}=2$ supersymmetric n-particles Euler-Calogero-Moser model admits a generalization to those which are invariant under the supersymmetry with arbitrary even number of supercharges forming the $\mathcal{N}=2M$ super Poincaré algebra. This generalization provides by fermions and supercharges which have the extra indices of the $su(M)$ algebra.

What is quite unexpected that in the present case the construction of $\mathcal{N}=2M$ supercharges $Q^a, \overline{Q}_a,\; a=1,M$ which form $\mathcal{N}=2M$ super Poincaré algebra

$$\left\{Q^a, \overline{Q}_b\right\} = -2\mathrm{i}\delta^a_b H, \quad \left\{Q^a, Q^b\right\} = \left\{\overline{Q}_a, \overline{Q}_b\right\} = 0, \tag{22}$$

goes straightforward. Indeed, consider the following set of $Mn(n+1)$ fermions $\psi^a_i, \bar\psi_{a\,i}$ and $\rho^a_{ij}, \bar\rho_{a\,ij}$ that satisfy the standard Poisson brackets

$$\left\{\psi^a_i, \bar\psi_{b\,j}\right\} = -\mathrm{i}\delta^a_b\delta_{ij}, \quad \left\{\rho^a_{ij}, \bar\rho_{b\,km}\right\} = -\frac{\mathrm{i}}{2}\delta^a_b\left(\delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk}\right). \tag{23}$$

Then, by analogy with the $\mathcal{N}=2$ supersymmetric case, it is possible to construct a composite object $\Pi_{ij}=-\Pi_{ji}$

$$\Pi_{ij}=-\mathrm{i}\left[\left(\psi_i^a-\psi_j^a\right)\bar\rho_{a\,ij}+\left(\bar\psi_{a\,i}-\bar\psi_{a\,j}\right)\rho^a_{ij}+\sum_{k=1}^{n}\left(\rho^a_{ik}\bar\rho_{a\,kj}-\rho^a_{jk}\bar\rho_{a\,ki}\right)\right], \tag{24}$$

that satisfies, as before, the commutation relations of the $so(n)$ algebra (17). Using (24), one can write the supercharges $Q^a, \overline{Q}_a$, which correspond to the extended $\mathcal{N}=2M$ supersymmetry, as follows

$$Q^a=\sum_{i=1}^{n}p_i\psi_i^a-\sum_{i\neq j}^{n}\frac{\left(\ell_{ij}+\Pi_{ij}\right)\rho^a_{ij}}{x_i-x_j},\quad \overline{Q}_a=\sum_{i=1}^{n}p_i\bar\psi_{a\,i}-\sum_{i\neq j}^{n}\frac{\left(\ell_{ij}+\Pi_{ij}\right)\bar\rho_{a\,ij}}{x_i-x_j} \tag{25}$$

Now one can check that they form together with the Hamiltonian

$$H=\frac{1}{2}\sum_{i=1}^{n}p_i^2+\sum_{i\neq j}^{n}\frac{\left(\ell_{ij}+\Pi_{ij}\right)^2}{2\left(x_i-x_j\right)^2}. \tag{26}$$

the $\mathcal{N}=2M$ super Poincaré algebra (22) and describe, therefore, the $\mathcal{N}=2M$ supersymmetric extension of the n-particles Euler-Calogero-Moser model.

$Osp(2M|2)$ superconformal symmetry of the model

The constructed $\mathcal{N} = 2M$ supersymmetric extension of the n-particles Euler-Calogero-Moser model described by (25) and (26) possesses a dynamical superconformal symmetry. It is rather easy to check that the supercharges Q^a, $\overline{Q}_a$ (25), Hamiltonian H (26) and the following conserved currents

$$
\begin{aligned}
&K = \frac{1}{2}\sum_{i=1}^{n} x_i^2 - t\sum_{i=1}^{n} x_i p_i + t^2 H, \quad D = -\frac{1}{2}\sum_{i=1}^{n} x_i p_i + tH, \\
&J^a{}_b = -\sum_{i=1}^{n} \psi_i^a \bar{\psi}_{b\,i} - \sum_{i\neq j}^{n} \rho_{ij}^a \bar{\rho}_{b\,ij}, \\
&I^{ab} = -\sum_{i=1}^{n} \psi_i^a \psi_i^b - \sum_{i\neq j}^{n} \rho_{ij}^a \rho_{ij}^b, \; \bar{I}_{ab} = \sum_{i=1}^{n} \bar{\psi}_{a\,i} \bar{\psi}_{b\,i} + \sum_{i\neq j}^{n} \bar{\rho}_{a\,ij} \bar{\rho}_{b\,ij}, \\
&S^a = \sum_{i=1}^{n} x_i \psi_i^a - tQ^a, \quad \overline{S}_a = \sum_{i=1}^{n} x_i \bar{\psi}_{a\,i} - t\overline{Q}_a
\end{aligned}
\tag{27}
$$

form the superalgebra $osp(2M|2)$. Here the generators $J^a{}_b$ form $u(M)$ subalgebra, while together with the generators I^{ab} and $\bar{I}_{ab}$ they form $so(2M)$ subalgebra of the superalgebra $osp(2M|2)$.

The (anti)commutation relations of generators of the $osp(2M|2)$ superalgebra read

$$
\begin{aligned}
&\{H,K\} = 2D, \quad \{H,D\} = H, \quad \{K,D\} = -K, \\
&\{J^a{}_b, J^c{}_d\} = \mathrm{i}\left(\delta^c_b J^a{}_d - \delta^a_d J^c{}_b\right), \left\{I^{ab}, \bar{I}_{cd}\right\} = \mathrm{i}\left(\delta^a_c J^b{}_d - \delta^a_d J^b{}_c - \delta^b_c J^a{}_d + \delta^b_d J^a{}_c\right), \\
&\left\{J^a{}_b, I^{cd}\right\} = \mathrm{i}\left(\delta^c_b I^{ad} - \delta^d_b I^{ac}\right), \left\{J^a{}_b, \bar{I}_{cd}\right\} = -\mathrm{i}\left(\delta^a_c \bar{I}_{bd} - \delta^a_d \bar{I}_{bc}\right), \\
&\{D, Q^a\} = -\frac{1}{2}Q^a, \left\{D, \overline{Q}_a\right\} = -\frac{1}{2}\overline{Q}_a, \quad \{D, S^a\} = \frac{1}{2}S^a, \left\{D, \overline{S}_a\right\} = \frac{1}{2}\overline{S}_a, \\
&\{H, S^a\} = -Q^a, \left\{H, \overline{S}_a\right\} = -\overline{Q}_a, \quad \{K, Q^a\} = S^a, \left\{K, \overline{Q}_a\right\} = \overline{S}_a, \\
&\{J^a{}_b, Q^c\} = \mathrm{i}\,\delta^c_b\, Q^a, \ \{J^a{}_b, S^c\} = \mathrm{i}\,\delta^c_b\, S^a, \\
&\left\{J^a{}_b, \overline{Q}_c\right\} = -\mathrm{i}\,\delta^a_c\, \overline{Q}_b, \ \left\{J^a{}_b, \overline{S}_c\right\} = -\mathrm{i}\,\delta^a_c\, \overline{S}_b, \\
&\left\{I^{ab}, \overline{Q}_c\right\} = -\mathrm{i}\left(\delta^a_c Q^b - \delta^b_c Q^a\right), \ \left\{I^{ab}, \overline{S}_c\right\} = -\mathrm{i}\left(\delta^a_c S^b - \delta^b_c S^a\right), \\
&\left\{\bar{I}_{ab}, Q^c\right\} = \mathrm{i}\left(\delta^c_a \overline{Q}_b - \delta^c_b \overline{Q}_a\right), \ \left\{\bar{I}_{ab}, S^c\right\} = \mathrm{i}\left(\delta^c_a \overline{S}_b - \delta^c_b \overline{S}_a\right), \\
&\left\{Q^a, \overline{Q}_b\right\} = -2\mathrm{i}\delta^a_b H, \ \left\{S^a, \overline{S}_b\right\} = -2\mathrm{i}\delta^a_b K, \\
&\left\{Q^a, \overline{S}_b\right\} = 2\,\mathrm{i}\,\delta^a_b\, D + J^a{}_b, \ \left\{S^a, \overline{Q}_b\right\} = 2\,\mathrm{i}\,\delta^a_b\, D - J^a{}_b, \\
&\left\{Q^a, S^b\right\} = I^{ab}, \ \left\{\overline{Q}_a, \overline{S}_b\right\} = -\bar{I}_{ab}.
\end{aligned} \tag{28}
$$

III. $\mathcal{N}=2$ Euler-Calogero-Moser model in superspace
Superfield constraints and action

To provide the superspace description of the n-particles $\mathcal{N}=2$ supersymmetric Euler-Calogero-Moser model, which defined by the supercharges $Q,\overline{Q}$

$$Q=\sum_{i=1}^{n} p_i\psi_i - \sum_{i\neq j}^{n}\frac{(\ell_{ij}+\Pi_{ij})\,\rho_{ij}}{x_i-x_j}\,, \qquad \overline{Q}=\sum_{i=1}^{n} p_i\bar\psi_i - \sum_{i\neq j}^{n}\frac{(\ell_{ij}+\Pi_{ij})\,\bar\rho_{ij}}{x_i-x_j}\,,$$

$$\Pi_{ij}=-\mathrm{i}\Big[(\psi_i-\psi_j)\,\bar\rho_{ij}+(\bar\psi_i-\bar\psi_j)\,\rho_{ij}+\sum_{k=1}^{n}(\rho_{ik}\bar\rho_{kj}-\rho_{jk}\bar\rho_{ki})\Big]$$

and the Hamiltonian

$$H=\frac{1}{2}\sum_{i=1}^{n}p_i^2+\sum_{i\neq j}^{n}\frac{(\ell_{ij}+\Pi_{ij})^2}{2\,(x_i-x_j)^2}\,,$$

one has to solve the following two tasks:

- one has to put the physical components $x_i,\psi_i,\bar\psi_i,\rho_{ij}$ and $\bar\rho_{ij}$ in the proper $\mathcal{N}=2$ superfields;
- it is necessary to introduce additional auxiliary bosonic $\mathcal{N}=2$ superfields $\boldsymbol{v}_i(t,\theta,\bar\theta)$ and $\bar{\boldsymbol{v}}_i(t,\theta,\bar\theta)$ in such a way that it will be possible to realize the $so(n)$ generators ℓ_{ij} through the bilinear combinations of their first bosonic components $v_i(t),\bar v_i(t)$ when passing to the Hamiltonian formalism.

Let us start with the first task.
It is completely clear from the structure of the supercharges $Q, \overline{Q}$ that under the $\mathcal{N}=2$ supersymmetry given by

$$\delta_{SUSY} z(t) = \mathrm{i}\left\{z(t), \bar{\varepsilon}Q + \varepsilon\overline{Q}\right\}_{PB}$$

the coordinates $x_i(t)$ transform through the fermions $\psi_i(t), \bar{\psi}_i(t)$.
Thus, one should introduce *n* bosonic $\mathcal{N}=2$ superfields $\boldsymbol{x}_i(t,\theta,\bar{\theta})$ with the following components

$$x_i = \boldsymbol{x}_i|, \quad \psi_i = -\mathrm{i}D\boldsymbol{x}_i|, \quad \bar{\psi} = -\mathrm{i}\overline{D}\boldsymbol{x}_i|, \quad A_i = \frac{1}{2}\left[\overline{D}, D\right]\boldsymbol{x}_i|. \tag{29}$$

Here, D and $\overline{D}$ are the $\mathcal{N}=2$ covariant derivatives obeying the relations

$$\left\{D, \overline{D}\right\} = 2\mathrm{i}\partial_t, \qquad \left\{D, D\right\} = \left\{\overline{D}, \overline{D}\right\} = 0,$$

and, as usually, $(...)|$ denotes the $\theta = \bar{\theta} = 0$ limit.

As concerning the fermionic components $\rho_{ij}(t), \bar\rho_{ij}(t)$, they can be placed in $n(n-1)$ fermionic superfields $\boldsymbol{\rho}_{ij}(t,\theta,\bar\theta), \bar{\boldsymbol{\rho}}_{ij}(t,\theta,\bar\theta)$ which are symmetric over the indices i,j and each have zero diagonal part

$$\boldsymbol{\rho}_{ij}=\boldsymbol{\rho}_{ji}, \quad \bar{\boldsymbol{\rho}}_{ij}=\bar{\boldsymbol{\rho}}_{ji}, \qquad \boldsymbol{\rho}_{ii}=\bar{\boldsymbol{\rho}}_{ii}=0, \quad \forall i.$$

Being $\mathcal{N}=2$ superfields, the $\boldsymbol{\rho}_{ij}(t,\theta,\bar\theta)$ and $\bar{\boldsymbol{\rho}}_{ij}(t,\theta,\bar\theta)$ contain a lot of components. However, the explicit form of the supercharges $Q,\overline{Q}$ leads to the following transformations of their lowest components under $\mathcal{N}=2$ supersymmetry

$$\delta_Q\rho_{ij} \sim \mathrm{i}\bar\epsilon\left[\frac{\psi_i-\psi_j}{x_i-x_j}\rho_{ij}-\sum_{k\neq i,j}^{n}\frac{x_i-x_j}{(x_i-x_k)(x_j-x_k)}\rho_{ik}\rho_{jk}\right],$$
$$\delta_{\overline{Q}}\bar\rho_{ij} \sim \mathrm{i}\epsilon\left[\frac{\bar\psi_i-\bar\psi_j}{x_i-x_j}\bar\rho_{ij}-\sum_{k\neq i,j}^{n}\frac{x_i-x_j}{(x_i-x_k)(x_j-x_k)}\bar\rho_{ik}\bar\rho_{jk}\right]. \qquad (30)$$

Thus, we have to impose the following nonlinear chirality conditions on superfields $\boldsymbol{\rho}_{ij}(t,\theta,\bar\theta)$ and $\bar{\boldsymbol{\rho}}_{ij}(t,\theta,\bar\theta)$

$$D\boldsymbol{\rho}_{ij} = \mathrm{i}\left[\frac{\boldsymbol{\psi}_i-\boldsymbol{\psi}_j}{\boldsymbol{x}_i-\boldsymbol{x}_j}\boldsymbol{\rho}_{ij} - \sum_{k\neq i,j}^{n}\frac{\boldsymbol{x}_i-\boldsymbol{x}_j}{(\boldsymbol{x}_i-\boldsymbol{x}_k)(\boldsymbol{x}_j-\boldsymbol{x}_k)}\boldsymbol{\rho}_{ik}\boldsymbol{\rho}_{jk}\right],$$

$$\overline{D}\bar{\boldsymbol{\rho}}_{ij} = \mathrm{i}\left[\frac{\bar{\boldsymbol{\psi}}_i-\bar{\boldsymbol{\psi}}_j}{\boldsymbol{x}_i-\boldsymbol{x}_j}\boldsymbol{\rho}_{ij} - \sum_{k\neq i,j}^{n}\frac{\boldsymbol{x}_i-\boldsymbol{x}_j}{(\boldsymbol{x}_i-\boldsymbol{x}_k)(\boldsymbol{x}_j-\boldsymbol{x}_k)}\bar{\boldsymbol{\rho}}_{ik}\bar{\boldsymbol{\rho}}_{jk}\right]. \quad (31)$$

These constraints leave in the superfields $\boldsymbol{\rho}_{ij}$ and $\bar{\boldsymbol{\rho}}_{ij}$ the following components

$$\rho_{ij} = \boldsymbol{\rho}_{ij}|, \quad B_{ij} = \overline{D}\boldsymbol{\rho}_{ij}|, \qquad \bar\rho_{ij} = \bar{\boldsymbol{\rho}}_{ij}|, \quad \overline{B}_{ij} = D\bar{\boldsymbol{\rho}}_{ij}|\,. \quad (32)$$

Finally, to have the proper brackets for $\psi_i, \bar\psi_i$ and $\rho_{ij}, \bar\rho_{ij}$, when passing to the Hamiltonian formalism, the kinetic terms for these fermionic components must have the form:

$$\mathcal{L}^{\psi}_{kin} = \frac{\mathrm{i}}{2}\sum_{i=1}^{n}\left(\dot\psi_i\bar\psi_i - \psi_i\dot{\bar\psi}_i\right), \quad \mathcal{L}^{\rho}_{kin} = \frac{\mathrm{i}}{2}\sum_{i,j}^{n}\left(\dot\rho_{ij}\bar\rho_{ij} - \rho_{ij}\dot{\bar\rho}_{ij}\right). \quad (33)$$

Combining all these together, we will come to the following superfield action for the purely $\mathcal{N}=2$ supersymmetric system (with $l_{ij}=0$)

$$S_0 = \int dt d^2\theta\left[-\frac{1}{2}\sum_{i=1}^{n} D\boldsymbol{x}_i\,\overline{D}\boldsymbol{x}_i + \frac{1}{2}\sum_{i,j}^{n}\boldsymbol{\rho}_{ij}\bar{\boldsymbol{\rho}}_{ij}\right], \qquad d^2\theta \equiv D\overline{D}. \quad (34)$$

Next task is to realize the operators ℓ_{ij} in terms of auxiliary semi-dynamical variables. Being the $so(n)$ generators, the operators ℓ_{ij} possess the standard realization in terms of n coordinates y_i and corresponding momenta p_i^y

$$\hat{\ell}_{ij} = \frac{1}{2}\left(y_i p_j^y - y_j p_i^y\right), \qquad \left\{y_i, p_j^y\right\} = \delta_{ij}. \tag{35}$$

However, such realization cannot be immediately implemented at the Lagrangian level. The way to get around this problem is to introduce new variables $v_i, \bar{v}_i$

$$v_i \equiv \frac{1}{\sqrt{2}}\left(y_i + \mathrm{i} p_i^y\right), \quad \bar{v}_i \equiv \frac{1}{\sqrt{2}}\left(y_i - \mathrm{i} p_i^y\right), \qquad \Rightarrow \quad \left\{v_i, \bar{v}_j\right\} = -\mathrm{i}\delta_{ij}, \tag{36}$$

and represent the generators $\hat{\ell}_{ij}$ as

$$\hat{\ell}_{ij} = \frac{\mathrm{i}}{2}\left(v_i \bar{v}_j - v_j \bar{v}_i\right). \tag{37}$$

To implement these semi-dynamical variables $v_i, \bar{v}_i$ at the superfield level we have to introduce $2n$ bosonic superfields $\boldsymbol{v}_i(t,\theta,\bar{\theta}), \bar{\boldsymbol{v}}_i(t,\theta,\bar{\theta})$. The additional information about these superfields comes from the transformations of $v_i, \bar{v}_i$ under the supersymmetry. These transformations can be learned from the explicit structure of supercharges $Q, \overline{Q}$ in which the operators ℓ_{ij} are replaced by their realization in terms of $v_i, \bar{v}_j$

$$\delta_Q v_i \sim \mathrm{i}\,\bar{\epsilon} \sum_{j\neq i}^{n} \frac{\rho_{ij} v_j}{x_i - x_j}, \qquad \delta_{\overline{Q}} \bar{v}_i \sim \mathrm{i}\,\epsilon \sum_{j\neq i}^{n} \frac{\bar{\rho}_{ij} \bar{v}_j}{x_i - x_j}. \tag{38}$$

Such form of the supersymmetry transformations means that the superfields $\boldsymbol{v}_i(t,\theta,\bar\theta)$ and $\bar{\boldsymbol{v}}_i(t,\theta,\bar\theta)$ are subjected the nonlinear chirality conditions

$$D\boldsymbol{v}_i = \mathrm{i}\sum_{j\neq i}^{n}\frac{\boldsymbol{\rho}_{ij}\boldsymbol{v}_j}{\boldsymbol{x}_i-\boldsymbol{x}_j}, \qquad \overline{D}\bar{\boldsymbol{v}}_i = \mathrm{i}\sum_{j\neq i}^{n}\frac{\bar{\boldsymbol{\rho}}_{ij}\bar{\boldsymbol{v}}_j}{\boldsymbol{x}_i-\boldsymbol{x}_j}. \tag{39}$$

These conditions leave in the superfields $\boldsymbol{v}_i(t,\theta,\bar\theta)$ and $\bar{\boldsymbol{v}}_i(t,\theta,\bar\theta)$ the components

$$v_i = \boldsymbol{v}_i|, \quad C_i = -\mathrm{i}\overline{D}\boldsymbol{v}_i|, \qquad \bar v_i = \bar{\boldsymbol{v}}_i|, \quad \overline{C}_i = -\mathrm{i}D\bar{\boldsymbol{v}}_i|. \tag{40}$$

It is quite clear that to have the standard Poisson brackets, the semi-dynamical variables $v_i, \bar v_i$ must have the following kinetic term in the Lagrangian

$$\mathcal{L}^{v}_{kin} = -\frac{\mathrm{i}}{2}\sum_{i=1}^{n}\left(\dot v_i\bar v_i - v_i\dot{\bar v}_i\right). \tag{41}$$

Thus, the interaction part of the superfield action reads

$$S_1 = -\frac{1}{2}\int dt d^2\theta \sum_{i=1}^{n}\boldsymbol{v}_i\bar{\boldsymbol{v}}_i. \tag{42}$$

Combining all these together, we conclude that the superfield action has the form

$$S = S_0 + S_1 = \int dt d^2\theta\Big[-\frac{1}{2}\sum_{i=1}^{n}D\boldsymbol{x}_i\,\overline{D}\boldsymbol{x}_i + \frac{1}{2}\sum_{i,j}^{n}\boldsymbol{\rho}_{ij}\bar{\boldsymbol{\rho}}_{ij} - \frac{1}{2}\sum_{i=1}^{n}\boldsymbol{v}_i\bar{\boldsymbol{v}}_i\Big], \tag{43}$$

where the superfields $\boldsymbol{\rho}_{ij}, \bar{\boldsymbol{\rho}}_{ij}, \boldsymbol{v}_i$ and $\bar{\boldsymbol{v}}_i$ are subjected to the constraints (31), (39).

Component action

Despite the extremely simple form of the superfield action (43), its component version looks quite complicated due to the nonlinear chirality constraints (31), (39). The corresponding component Lagrangian can be written as a sum of the kinetic terms $\mathcal{L}_{kin}$, the Lagrangians containing the corresponding auxiliary fields $\mathcal{L}^A_{Aux}$, $\mathcal{L}^B_{Aux}$, $\mathcal{L}^C_{Aux}$ and the "matter" Lagrangian $\mathcal{L}_{matter}$

$$\mathcal{L} = \mathcal{L}_{kin} + \mathcal{L}^A_{Aux} + \mathcal{L}^B_{Aux} + \mathcal{L}^C_{Aux} + \mathcal{L}_{matter}\,. \tag{44}$$

Eliminating the auxiliary fields $A_i, B_{ij}, \overline{B}_{ij}, C_i, \overline{C}_i$, using their equations of motion, we obtain the component on-shell Lagrangian

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\sum_{i=1}^{n} \dot{x}_i\dot{x}_i + \frac{\mathrm{i}}{2}\sum_{i=1}^{n}\left(\dot{\psi}_i\bar{\psi}_i - \psi_i\dot{\bar{\psi}}_i\right) + \frac{\mathrm{i}}{2}\sum_{i,j}^{n}\left(\dot{\rho}_{ij}\bar{\rho}_{ij} - \rho_{ij}\dot{\bar{\rho}}_{ij}\right) \\ &- \frac{\mathrm{i}}{2}\sum_{i=1}^{n}\left(\dot{v}_i\bar{v}_i - v_i\dot{\bar{v}}_i\right) - \frac{1}{2}\sum_{i\neq j}^{n}\frac{\left(\hat{\ell}_{ij} + \Pi_{ij}\right)^2}{(x_i - x_j)^2}. \end{aligned} \tag{45}$$

Thus, the superfield action (43) with the superfields $\boldsymbol{\rho}_{ij}, \bar{\boldsymbol{\rho}}_{ij}, \boldsymbol{v}_i$ and $\bar{\boldsymbol{v}}_i$ subjected to the nonlinear chirality constraints (31) and (39) indeed describes the $\mathcal{N} = 2$ supersymmetric n-particles Euler-Calogero-Moser model.

IV. Concluding remarks

To conclude, let us make a few comments:

- The nonlinear chirality conditions (31) can be slightly simplified by passing to the superfields $\xi_{ij}, \bar{\xi}_{ij}$:

$$\xi_{ij} \equiv \frac{\rho_{ij}}{\mathbf{x}_i - \mathbf{x}_j}, \quad \bar{\xi}_{ij} \equiv \frac{\bar{\rho}_{ij}}{\mathbf{x}_i - \mathbf{x}_j} \quad \Rightarrow \quad D\xi_{ij} + \mathrm{i}\sum_{k=1}^{n} \xi_{ik}\xi_{jk} = 0, \;\; \overline{D}\bar{\xi}_{ij} + \mathrm{i}\sum_{k=1}^{n} \bar{\xi}_{ik}\bar{\xi}_{jk} = 0.$$

  However, the Lagrangian, Hamiltonian and the Poisson brackets will look more complicated, being written through the variables ξ_{ij} and $\bar{\xi}_{ij}$, in spite of the fact that these superfields are now defined independently of the superfields $\mathbf{x}_i$.

- It turns out that the auxiliary superfields $\mathbf{v}_i, \bar{\mathbf{v}}_i$ cannot be redefined in a similar manner. Thus, the nonlinear chirality constraints (39) which relate these superfields with the $\mathbf{x}_i$ are crucial for the superfields description.

- It should be noted that the semi-dynamical variables $v_i, \bar{v}_i$ obeying the brackets (36) can be used for the construction of $su(n)$ generators. Clearly, the kinetic Lagrangian $\mathcal{L}^{v}_{kin}$ (41) possesses $su(n)$ symmetry. However, this $su(n)$ symmetry is reduced to the $so(n)$ upon using the nonlinear chirality constraints (39).