BRST approach for continuous spin and massive higher spin field theory

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Based on discussion with I.L. Buchbinder and V. Krykhtin

Plan of talk

I. Introduction Continuous spin field

1-1. History

- Found by Bargmann and Wigner, Proc. Nat. Acad. Sci. 34(1948) 211.
- Lagrangian is firstly presented by Schuster and Toro, PRD 91 (2015).
- Higher/lower dimension, fermionic case and relation massive HS or string theory are analised by Metsaev, Zinoviev, Khabarov, Bekaert, Skvortsov, Najafizadeh, Mourad, Setare, Savvidy and so on.
- Lagrangian by BRST construction is studied by I.L. Buchbinder, V.A. Krykhtin, H.T, to be appeared in PLB, arXiv:1806.01640

1-2. Special interest for this talk

- Continuous spin limit from Massive HS, Bekaert and Mourad, JHEP0601:115,2006
- Lagrangian having two mass dimension full parameter m and μ by Metsaev, PLB 767(2017),PLB781(2018)

I. Introduction Casimir condition

1-3.Irreducible representatoin of Poincare algebra

Two dimension-full parameters m and μ classify irreducible representation

1. Massive spin *s* field : m≠0 and μ=0.

$$P^2\Psi = m^2\Psi \qquad W^2\Psi = m^2 s(s+1)\Psi$$

2. Continuous spin: m=0 and μ≠0.

$$P^2\Psi = 0 \qquad W^2\Psi = \mu^2\Psi$$

Formulation depends on representation of Poincare algebra

Two cases are shown in this talk

1. Tensor with vector index (SO(3,1) index)
2. Tensor with spinor index(SL(2,C) index)

II. Continousoun spin field with vector index

2-1.Poincare algebra and Casimir

2-1-1. Poincare algebra

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i\eta^{\mu\rho}M^{\nu\sigma} - \mu \leftrightarrow \nu - \rho \leftrightarrow \sigma$$

$$[M^{\mu\nu}, P^{\lambda}] = -i\eta^{\lambda\mu}P^{\nu} - \mu \leftrightarrow \nu \ , \quad [P^{\mu}, P^{\nu}] = 0$$

Representation by space-time coordinate and oscillators on Fock space:

$$M_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu} \qquad L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$$

$$P_\mu = i\partial_\mu \qquad S_{\mu\nu} = i(a_\mu^\dagger a_\nu - a_\nu^\dagger a_\mu)$$

$$|\psi\rangle = \sum_{s=0}^{\infty} a^{\mu_1 \dagger} \cdots a^{\mu_s \dagger}\psi_{\mu_1\cdots\mu_s} \qquad [a_\mu, a_\nu^\dagger] = \eta_{\mu\nu}$$

2-1-2. Casimir operators

1st Casimir $C_1 = P^2$ is in this representation

$$P^2|\psi\rangle = -\partial^2|\psi\rangle$$

2nd Casimir $C_2 = W^2$ is Pauli-Lubanski vector^2 in this representation

$$W^2|\psi\rangle = \left[-(\partial_\nu S^{\nu\lambda})(\partial^\mu S_{\mu\lambda}) + \tfrac{1}{2}\partial^2 S^{\nu\lambda}S_{\nu\lambda}\right]|\psi\rangle$$

II. Continuous spin field with vector index

2-2.Constraints for irredcibility for continuous spin field

Wigner's equation for continuous spin represented by oscillators

$$
\begin{aligned}
p^2|\psi\rangle &= 0 && \text{...KG equation(mass less)} \\
(a^{\dagger\lambda}p_\lambda + \mu)|\psi\rangle &= 0 \\
a^\lambda p_\lambda|\psi\rangle &= 0 && \text{...Divergence free condition} \\
(a^\lambda a_\lambda + 1)|\psi\rangle &= 0 && \text{...Trace condition}
\end{aligned}
$$

$$p_\mu := -i\partial_\mu$$

$$|\psi\rangle = \sum_{s=0}^{\infty} a^{\dagger\mu(s)}\psi_{\mu(s)}$$

↓ Under these conditions

$$
\begin{aligned}
P^2|\psi\rangle &= 0 \\
W^2|\psi\rangle &= \mu^2|\psi\rangle
\end{aligned}
$$

ψ is Irreducible representation of continuous spin field.

II. Continousoun spin field with vector index

2-3.Solusion

$$\begin{aligned}\partial^2|\psi\rangle &= 0\\ (a^\dagger\cdot p+\mu)|\psi\rangle &= 0\\ a\cdot p|\psi\rangle &= 0\\ (a^2+1)|\psi\rangle &= 0\end{aligned}$$

2nd condition is solved as

$$|\psi\rangle = \delta(a^{\dagger\lambda}p_\lambda+\mu)|\tilde\psi\rangle$$

$$\delta(\hat{\mathcal{O}}) := \int \frac{dk}{2\pi} e^{ik\hat{\mathcal{O}}}$$

We get equivalent set of conditions:

$$\begin{aligned}p^2|\tilde\psi\rangle &= 0\\ a^\lambda p_\lambda|\tilde\psi\rangle &= 0\\ (a^\lambda a_\lambda+1)|\tilde\psi\rangle &= 0\end{aligned}$$

$$|\tilde\psi(x)\rangle = \sum_{s=0}^{\infty} a^{\mu_1\dagger}\cdots a^{\mu_s\dagger}|0\rangle\tilde\psi_{\mu_1\cdots\mu_s}(x)$$

Closed HS algebra for these constraints is not known

Difficulty is from constant term of trace condition.
Bypass from "massive" case, using continuous spin limit of Bekaert and Skvortsov

II. Continousoun spin field with vector index

2-4.Continuous spin limit from massive HS field

A possible way to avoid the difficulty: Continuous spin limit from massive HS field

Ref: “The continuous spin limit of higher spin field equations”, JHEP0601:115,2006, Bekaert and Skvortsov

continuous spin limit: $m \to 0, s \to \infty, ms = \mu:$ fixed

We will translate the procedure in oscillator language

2-4-1. Oscillator realization for massive spin s field

$$\begin{aligned} L_0|\Phi\rangle &= 0 \\ L_1|\Phi\rangle &= 0 \\ L_2|\Phi\rangle &= 0 \\ (N_a + N_b - s)|\Phi\rangle &= 0 \end{aligned}$$

spin fixing condition

$$p_\mu := -i\partial_\mu$$

$$\begin{aligned} L_0 &= p^2 - m^2 \\ L_1 &= p^\lambda a_\lambda + mb \\ L_1^\dagger &= p^\lambda a_\lambda^\dagger + mb^\dagger \\ 2L_2 &= a^\lambda a_\lambda + b^2 \\ N_a &= a^{\dagger\lambda} a_\lambda \\ N_b &= b^\dagger b \end{aligned}$$

Closed HS algebra can be found for HS operators.

(NPB727 (2005) 537, Buchbinder, Krykhtin)

II. Continousoun spin field with vector index

2-4.Continuous spin limit from massive HS field

2-4-2. new oscillator and vacuum

$$c^\lambda := \frac{1}{s} a^\lambda b^\dagger \qquad |0\rangle_s := \frac{b^{\dagger s}}{\sqrt{s!}}|0\rangle$$

$$c^{+\lambda} := a^{\dagger\lambda} b \qquad c^\lambda |0\rangle_s = {}_s\langle 0| c^{+\lambda} = 0$$

spin fixing condition $(N_a + N_b - s)|\Phi\rangle = 0$ is solved as

$$|\Phi\rangle_s = \sum_{n=0}^{s} (c^+)^{\mu(n)} |0\rangle_s \bar{\Phi}_{\mu(n)}$$

Other constraint equations are rewritten as

$$0 = L_0 |\Phi\rangle_s = \sum_{n=0}^{s} \left(p^2 + \frac{\mu^2}{s^2} \right) (c^+)^{\mu(n)} |0\rangle_s \sqrt{s!} \bar{\Phi}_{\mu(n)}$$

$$0 = L_1 |\Phi\rangle_s = \sum_{n=0}^{s} \left(p_\lambda c^\lambda + \frac{\mu}{s} \left(1 - \frac{n}{s} \right) \right) (c^+)^{\mu(n)} |0\rangle_{s-1} \frac{s\sqrt{(s-1)!}}{1 - \frac{n}{s}} \bar{\Phi}_{\mu(n)}$$

$$0 = L_2 |\Phi\rangle_s = \sum_{n=0}^{s} \left(c^\lambda c_\lambda + \left(1 - \frac{n}{s} \right) \left(1 - \frac{1}{s} - \frac{n}{s} \right) \right) (c^+)^{\mu(n)} |0\rangle_{s-2} \frac{\frac{s^2}{2}\sqrt{(s-2)!}}{\left(1 - \frac{n}{s}\right)\left(1 - \frac{n}{s} - \frac{1}{s}\right)} \bar{\Phi}_{\mu(n)}$$

II. Continousoun spin field with vector index

2-4.Continuous spin limit from massive HS field

2-4-3.Continuopus spin equation

By taking limit: $m \to 0, s \to \infty, \mu := ms$ fixed We get

$$\begin{aligned} p^2 |\Phi\rangle_\infty &= 0 \\ p^\lambda c_\lambda |\Phi\rangle_\infty &= 0 \\ (c^\lambda c_\lambda + 1)|\Phi\rangle_\infty &= 0 \end{aligned}$$

$$\begin{aligned} |\Phi\rangle_\infty &= \textstyle\sum_{n=0}^{s} (c^+)^{\mu(n)} |0\rangle_\infty \bar{\Phi}_{\mu(n)} \\ [c_\mu, c_\nu^+] &= \eta_{\mu\nu} \end{aligned}$$

Continuous spin equations are derived

2-5.Stuckrelberg-Fronsdal Lagrangian for massive HS

$$\mathcal{L}_s =_s \langle\varphi|(L_0 - L_1^+ L_1 - L_2^\dagger(2L_0 + L_1^\dagger L_1)L_2 + L_2^\dagger L_1^2 + L_1^{+2} L_2)|\varphi\rangle_s$$

$$\delta|\varphi\rangle_s = L_1^\dagger|\lambda\rangle_{s-1}$$

Unfortunately, terms in Lagrangian does not have the same order of s.
This limit does not lead to Lagrangian for continuous spin.

III. Continousoun spin field by spinor index

Krykhtin's talk or PLB(to be appeared), e-print: arXiv:1806.01640, I.L. Buchbinder, V.A. Krykhtin, H. T. "BRST approach to Lagrangian construction for bosonic continuous spin field"

3-1. Lagrangian for continous spin field by BRST method

3-1-1. Oscillator representation

*space-time dimension=4

$$a, \dot{a} = 1, 2$$

$$[a_a, c^b] = \delta_a^b, [\bar{a}^{\dot{b}}, \bar{c}_{\dot{a}}] = \delta_{\dot{a}}^{\dot{b}}$$

$$a_a^\dagger = \bar{c}_{\dot{a}}, \bar{a}_{\dot{a}}^\dagger = c_a$$

$$|\varphi\rangle = \sum_{n,k} c^{a(k)} \bar{c}^{\dot{a}(l)} |0\rangle \frac{1}{\sqrt{n!k!}} \varphi_{a(k),\dot{a}(l)}$$

$$S_{\mu\nu} = -\frac{1}{2} c_a (\sigma^{\mu\nu})^{ab} a_b + \frac{1}{2} \bar{c}_{\dot{a}} (\bar{\sigma}^{\mu\nu})^{\dot{a}\dot{b}} \bar{a}_{\dot{b}}$$

$$l_{+a} = \bar{a}_{\dot{a}} i \bar{\partial}^{\dot{a}b} a_b$$

$$l_{-a} = l_a^\dagger \qquad \bar{\partial} = \bar{\sigma}^\mu \partial_\mu$$

3-1-1. Constraints for continuous spin

$$\begin{aligned} \partial^2 |\psi\rangle &= 0 \\ (l_{+a} - \mu)|\psi\rangle &= 0 \\ |\varphi\rangle = \delta(l_{-a} - \mu)|\psi\rangle & \end{aligned}$$

$$\Rightarrow$$

$$P^2|\varphi\rangle = 0$$

$$W^2|\varphi\rangle = \mu^2|\varphi\rangle$$

Continuous spin

III. Continousoun spin field by spinor index

3-1. Lagrangian for continous spin field by BRST method

3-1-1. Continusou spin algebra

$$
\begin{aligned}
[l_{+a}, l_{-a}] &= N_+\partial^2, \quad N_+ &= N_a + \bar{N}_a + 2 \\
[N_+, l_{\pm a}] &= \mp 2 l_{\pm a} \qquad\qquad N_a &= c^a a_a \\
& \qquad\qquad\qquad\qquad \bar{N}_a &= \bar{c}_a \bar{a}^a
\end{aligned}
$$

These form 1st class constraints algebra:

3-1-2. Lagrangian and gauge transformation

Lagrangian

$$
\mathcal{L} = \begin{pmatrix} \langle s_1| & \langle s_2| & \langle a_1| \end{pmatrix} \begin{pmatrix} \partial^2 & 0 & -(l_{-a}-\mu) \\ 0 & -\partial^2 & -(l_{+a}-\mu) \\ -(l_{+a}-\mu) & -(l_{-a}-\mu) & N_+ \end{pmatrix} \begin{pmatrix} |s_1\rangle \\ |s_2\rangle \\ |a_1\rangle \end{pmatrix}
$$

Gauge transformation

$$
\begin{aligned}
\delta|s_1\rangle &= (l_{-a}-\mu)|\lambda_1\rangle \\
\delta|s_2\rangle &= -(l_{+a}-\mu)|\lambda_1\rangle \\
\delta|a_1\rangle &= \partial^2|\lambda_1\rangle
\end{aligned}
$$

remaining condition →

$$
\delta(\delta(l_{-a}-\mu)|s_1\rangle) = 0
$$

IV. Mass defomation of continuous spin theory

Ref: Lagrangian having two mass dimension full parameters by Metsaev, PLB 767(2017), PLB781(2018)

no irr. rep. with $m\mu \neq 0$

4-1. Goal and strategy

Goal: Find consistent theory connecting
the continuous spin theory and a massive theory.

Strategy

- Deform continuous spin algebra by mass. ~~Casimir condition~~
- Find closed algebra with 1st class constraints.
- Constraint must reproduce continuous spin equations in $m \to 0$

IV. Mass defomation of continuoous spin theory

4-2. HS algebra and constraint equations

We introduce additional oscillators just as copy of original ones

$$[a_a, c^b] = \delta_a^b \quad , \quad [b_a, d^b] = \delta_a^b$$
$$[\bar{a}^{\dot{b}}, \bar{c}_{\dot{a}}] = \delta_{\dot{a}}^{\dot{b}} \quad , \quad [\bar{b}^{\dot{a}}, \bar{d}_{\dot{b}}] = \delta_{\dot{b}}^{\dot{a}}$$

c's and d's are creation op.

We prepare two set of mas less operators: $l_{\pm a}$ and $l_{\pm b}$ and interference term $m_\pm$, then define new op. $L_\pm$:

$$L_0 = \partial^2 + m^2 \quad , \quad m_+ := -m(b^a a_a + \bar{a}_{\dot{a}} \bar{b}^{\dot{a}})$$
$$L_\pm = l_{\pm a} + l_{\pm b} + m_\pm \quad , \quad m_- := -m(d^a c_a + \bar{c}_{\dot{a}} \bar{d}^{\dot{a}})$$

We can find the same constraint algebra to continuous spin.

HS algebra:

$$[L_+, L_-] = \mathbf{N}_+ L_0$$
$$[\mathbf{N}_+, L_\pm] = \mp 2 L_{\pm 1}$$

$$\mathbf{N}_+ := \mathbf{N} + \bar{\mathbf{N}} + 4$$
$$\mathbf{N} := N_a + \bar{N}_b$$
$$\bar{\mathbf{N}} := \bar{N}_a + N_b$$

Constraint(candidate)

$$L_0 |\psi\rangle = 0$$
$$(L_+ - \mu)|\psi\rangle = 0$$

$m \to 0$

$b|\psi\rangle = \bar{b}|\psi\rangle = 0$

$$\partial^2 |\psi\rangle = 0$$
$$(l_a - \mu)|\psi\rangle = 0$$

: Continuous spin

Closed algebra is found as similar to massless case. Constraints are 1st class

IV. Mass defomation of continuoous spin theory

4-3. Fock state

$$|\psi\rangle = \sum_{k,k',l,l'=0}^{\infty} c^{a(k')}\bar{c}_{\dot{a}(l')}d^{b(l)}\bar{d}_{\dot{b}(k)}|0\rangle \frac{(k+k')!}{k!k'!}\frac{(l+l')!}{l!l'!}\psi^{\dot{b}(k),\dot{a}(l')}_{a(k'),b(l)}$$

$(\mathbf{N}-\bar{\mathbf{N}})|\psi\rangle = 0$ Totally symmetric bosonic tensor fields(in vector index)

4-4. BRST operator and extended Fock state

Nil-potent BRST operator Q

$$Q = \eta_0 L_0 - P_0\eta_1^\dagger\eta_1 N_+ + \Delta Q \qquad Q^2 = 0$$

$$\Delta Q = \eta_1(L_- - \mu) + \eta_1^\dagger(L_+ - \mu)$$

Extended states

Dynamical fields are identified with state with ghost number =0. It is expanded by η_0.

$$\begin{aligned} |\chi^0\rangle &= |s_1\rangle + P_1^\dagger\eta_1^\dagger|s_2\rangle + \eta_0 P_1^\dagger|a_1\rangle \\ \langle\chi^0| &= \langle s_1| + \langle s_2|\eta_1 P_1 + \langle a_1|P_1\eta_0 \end{aligned}$$

Gauge parameter for the above dynamical field are identified with state with ghost number =-1:

$$|\chi^{-1}\rangle = P_1^\dagger|\lambda_1\rangle$$

s,a,λ are has polynomial like ψ

IV. Mass defomation of continuoous spin theory

4-5. Lagrangian and gauge transformation for massive

Lagrangian

$$\mathcal{L} = \int d\eta_0 \langle \chi^0 | Q | \chi^0 \rangle$$

$$= \begin{pmatrix} \langle s_1 | & \langle s_2 | & \langle a_1 | \end{pmatrix} \begin{pmatrix} L_0 & 0 & -(L_- - \mu) \\ 0 & -L_0 & -(L_+ - \mu) \\ -(L_+ - \mu) & -(L_- - \mu) & \mathbf{N}_+ \end{pmatrix} \begin{pmatrix} |s_1\rangle \\ |s_2\rangle \\ |a_1\rangle \end{pmatrix}$$

Gauge transformation

$$\delta|s_1\rangle = (L_- - \mu)|\lambda_1\rangle \,, \quad \delta|s_2\rangle = -(L_+ - \mu)|\lambda_1\rangle \,, \quad \delta|a_1\rangle = L_0|\lambda_1\rangle$$

Note: massless limit

- Even after setting m=0, there is larger gauge d.o.e. than that of massless case.
- Continuous spin field(m=0) for $|s_1\rangle$ still has gauge degree of freedom:

$$\delta\{\delta(l_{-a} - \mu)|s_1\rangle\} = l_{-b}|\lambda_1\rangle$$

V. Summary and Tasks

Studied

- Winger's constraint equation for continuous spin field are represented by using oscillators with vector or spinor indexes.
- Constraint of massive HS is shown to become that of continuous spin field in a continuous spin limit.
- Mass deformation of continuous spin algebra and Langrangisn are presented.

Tasks

- Eliminate auxiliary fields in massive model and find constraints.
- Relate massive HS theory between vector and spinor index formulation.
- Reproduce continuous spin Lagrangian from massive one.
- Relate Lagrangian having two mass dimension full parameters to known models.
- Fermionic case and supersymmetric case.