From Coxeter Higher-Spin Theories
to
Strings and Tensor Models

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Plan

- Introduction: HS theory versus Strings
- HS modules and difficulty of the naive extension of HS theory
- Framed oscillator algebra
- Nonlinear HS equations
- Coxeter groups and Cherednik algebras
- Framed Cherednik systems
- Coxeter HS equations
- Relation with strings and tensor models
- Conclusion

Challenge: Quantum Gravity and String Theory

Conjecture: trans-Planckian regime exhibits high symmetries

D. Gross 1988, MV 1987...

Key idea of HS gauge theory: to understand what higher symmetries are possible

Important feature: $(A)dS$ **background with** $\Lambda \neq 0$ Fradkin, MV, 1987

HS theories: $\Lambda \neq 0$, $m = 0$, **symmetric fields** $s = 0, 1, 2, \ldots \infty$

First Regge trajectory

String Theory: $\Lambda = 0$, $m \neq 0$ **except for a few zero modes**

Infinite set of Regge trajectories

What is a HS symmetry of a string-like extension of HS theory?

MV 2012, Gaberdiel and Gopakumar 2014-2018

String Theory as spontaneously broken HS theory?! $(s > 2, m > 0)$

HS Algebra and Modules

Free field analysis: realization of the HS algebra hs_1 as Weyl algebra

$$[y_\alpha\,, y_\beta]_* = 2i\varepsilon_{\alpha\beta}\,, \qquad [\bar{y}_{\dot\alpha}\,, \bar{y}_{\dot\beta}]_* = 2i\varepsilon_{\dot\alpha\dot\beta}$$

Fradkin, MV 1987

AdS_4 algebra $sp(4) \sim o(3,2)$

Naive way to extend the spectrum of fields $y_\alpha \to y_\alpha^n$

does not lead to physically acceptable HS theories

The Fock hs_1-module F_1 describes free boundary conformal fields

$$D|0\rangle = h_1|0\rangle$$

Lowest weight representations of the naively extended algebras hs_p

built from p copies of oscillators have too high weights

$$h_p = ph_1$$

$F_1 \otimes F_1$ = massless fields in the bulk   Flato, Fronsdal (1978)

For $p > 1$ the lowest weights in $F_p \otimes F_p$ have no room for gravity

(massless spin-two)

Framed Oscillator Algebras

The problem is resolved in the framed oscillator algebras replacing usual oscillator algebra

$$[y^n_\alpha , y^m_\beta]_* = 2i\delta^{nm}\epsilon_{\alpha\beta}I \,,$$

where I is the unit element by

$$[y^n_\alpha , y^m_\beta]_* = 2i\delta^{nm}\epsilon_{\alpha\beta}I_n$$

"Units" I_n are assigned to each specie of the oscillators forming a set of commutative central idempotents

$$I_iI_j = I_jI_i \,, \qquad I_iI_i = I_i$$

This allows us to consider Fock modules F_i obeying

$$I_jF_i = \delta_{ij}F_i$$

equivalent to those of the single-oscillator case

Nonlinear HS Equations

HS star product

$$(f*g)(Z,Y) = \int dSdT \, \exp iS_A T^A \, f(Z+S, Y+S) g(Z-T, Y+T)$$

$[Y_A, Y_B]_* = -[Z_A, Z_B]_* = 2iC_{AB}\,,$ $Z-Y \,:\, Z+Y$ **normal ordering**

Inner Klein operators:

$\kappa = \exp iz_\alpha y^\alpha\,,$ $\bar\kappa = \exp i\bar z_{\dot\alpha}\bar y^{\dot\alpha}\,,$ $\kappa * f = \tilde{f} * \kappa\,,$ $\kappa*\kappa = 1$

$$\begin{cases} \mathrm{d}_x W + W*W = 0 \\ \mathrm{d}_x B + W*B - B*W = 0 \\ \mathrm{d}_x S + W*S + S*W = 0 \\ \mathbf{S*B - B*S = 0} \\ \mathbf{S*S = i(dZ^A dZ_A + \eta dz^\alpha dz_\alpha B*k*\kappa + \bar\eta d\bar z^{\dot\alpha} d\bar z_{\dot\alpha} B*\bar k*\bar\kappa)} \end{cases}$$

1992

Dynamical content is located in the x-independent twistor sector

The non-zero curvature has the form of Z_2-Cherednik algebra

Coxeter Groups and Cherednik Algebras

A rank-p Coxeter group $\mathcal{C}$ is generated by reflections with respect to a system of root vectors $\{v_a\}$ in a p-dimensional Euclidean vector space V. An elementary reflection associated with the root vector v_a

$$R_{v_a} x^i = x^i - 2 v_a^i \frac{(v_a, x)}{(v_a, v_a)}\,, \qquad R_{v_a}^2 = I$$

Cherednik deformation of the semidirect product of the oscillator algebra with the group algebra of $\mathcal{C}$ is

$$[q_\alpha^n\,, q_\beta^m] = -i\epsilon_{\alpha\beta}\left(2\delta^{nm} + \sum_{v\in\mathcal{R}} \nu(v)\frac{v^n v^m}{(v,v)} k_v\right)\,, \qquad k_v q_\alpha^n = R_v{}^n{}_m q_\alpha^m k_v$$

q_α^n $(\alpha = 1, 2,\ n = 1; \ldots, p)$

Coupling constants $\nu(v)$ are invariants of $\mathcal{C}$ being constant on the conjugacy classes of root vectors under the action of $\mathcal{C}$.

Double commutator of q_α^n respects Jacobi identities.

B_p–Coxeter System

Important case of the Coxeter root system is B_p with the roots

$$R_1 = \{\pm e^n \qquad 1 \le n \le p\}\,, \qquad R_2 = \{\pm e^n \pm e^m \quad 1 \le n < m \le p\}\,.$$

Apart from permutations B_p contains reflections of basis axes $v^n_{\pm} = e^n$.

R_1 and R_2 form two conjugacy classes of B_p.

The Coxeter group of $3d$ HS theory is $A_1 \sim B_1$.

B_2 underlies the string-like HS models.

The fact of fundamental importance for HS theories is that for any Coxeter root system the generators

$$t_{\alpha\beta} := \frac{i}{4} \sum_{n=1}^{p} \{q^n_\alpha\,, q^n_\beta\}$$

obey the $sp(2)$ commutation relations properly rotating all indices α

$$[t_{\alpha\beta}\,, q^n_\gamma] = \epsilon_{\beta\gamma} q^n_\alpha + \epsilon_{\alpha\gamma} q^n_\beta$$

Framed Cherednik Systems

A_{p-1} **system.** **In addition to** $q_{\alpha n}$ **and** k_{nm}, $n,m=1,\dots p$ **introduce** I_n

$$I_n I_m = I_m I_n\,, \qquad I_n I_n = I_n\,, \qquad I_n q_{\alpha n} = q_{\alpha n} I_n = q_{\alpha n}\,, \qquad I_n q_{\alpha m} = q_{\alpha m} I_n\,.$$

In presence of I_n **the deformed oscillator relations respecting Jacobi**

$$[q_{\alpha\,n}\,,q_{\beta\,m}] = -i\epsilon_{\alpha\beta}\Big(\delta_{nm}\Big(2I_n+\nu\sum_{l=1}^{p}\widehat{k}_{ln}\Big)-\nu\widehat{k}_{nm}\Big)\,, \qquad \widehat{k}_{nm}=I_nI_mk_{nm}\,.$$

$\widehat{k}_{nm}$ **obey all relations of** S_p **except for involutivity replaced by**

$$\widehat{k}_{nm}\widehat{k}_{nm}=I_nI_m\,.$$

$$I_l\widehat{k}_{nm}=\widehat{k}_{nm}I_l \quad \forall\; l,n,m\,, \qquad I_n\widehat{k}_{nm}=I_m\widehat{k}_{nm}=\widehat{k}_{nm}\,.$$

General Framed Cherednik Algebra

$$[q^n_\alpha\,,q^m_\beta] = -i\epsilon_{\alpha\beta}\left(2\delta^{nm}I_n+\sum_{v\in\mathcal{R}}\nu(v)\frac{v^nv^m}{(v,v)}\widehat{k}_v\right)\,, \qquad \widehat{k}_v := k_v\prod I_{i_1(v)}\dots I_{i_k(v)}$$

Framed Cherednik algebra still possesses inner $sp(2)$ **automorphisms**

$$t_{\alpha\beta} := \frac{i}{4}\sum_{n=1}^{p}\{q^n_\alpha\,,q^n_\beta\}I_n$$

Framed Star Product

x-dependent fields W, S and B depend on p sets of variables Y^n_A, Z^n_A $(A = 1, \dots M)$, I_n, anticommuting differentials dZ^A_n $(n = 1, \dots p)$ and Klein-like operators $\hat{k}_v$ associated with all roots of $\mathcal{C}$. Coxeter HS field equations are formulated in terms of the star product

$$(f*g)(Z;Y;I) = \frac{1}{(2\pi)^{pM}} \int d^{pM}S\, d^{pM}T \exp\left[iS^A_n T^B_m \delta^{nm} C_{AB}\right] f(Z_i + I_i S_i; Y_i + I_i S_i; I)$$

$$I_n * Y^n_A = Y^n_A * I_n = Y^n_A\,, \qquad I_n * Z^n_A = Z^n_A * I_n = Z^n_A\,, \qquad I_n * I_n = I_n$$

Implying

$$[Y^n_A, Y^m_B]_* = -[Z^n_A, Z^m_B]_* = 2iC_{AB}\delta^{nm} I_n\,, \qquad [Y^n_A, Z^m_B]_* = 0\,.$$

This star product admits inner Coxeter-Klein operators

$$\exp i \frac{v^n v^m Z_{\alpha n} Y^{\alpha}{}_m}{(v,v)}$$

Coxeter HS Equations

Unfolded equations for $\mathcal{C}$-HS theories remain the same except for

$$iS*S = dZ^{An}dZ_{An} + \sum_i \sum_{v\in\mathcal{R}_i} F_{i*}(B)\frac{dZ_n^{\alpha}v^n dZ_{\alpha\,m}v^m}{(v,v)} * \kappa_v$$

κ_v are generators of $\mathcal{C}$ acting trivially on all elements except for $dZ_{\alpha n}$

$$\kappa_v * dZ_\alpha^n = R_v{}^n{}_m dZ_\alpha^m * \kappa_v$$

$F_{i*}(B)$ is any star-product function of the zero-form B on the conjugacy classes R_i of $\mathcal{C}$. In the important case of the Coxeter group B_p

$$iS*S = dZ_{An}dZ^{An} + \sum_{v\in\mathcal{R}_1} F_{1*}(B)\frac{dZ_n^{\alpha}v^n dZ_{\alpha\,m}v^m}{(v,v)} * \kappa_v + \sum_{v\in\mathcal{R}_2} F_{2*}(B)\frac{dZ_n^{\alpha}v^n dZ_{\alpha\,m}v^m}{(v,v)} * \kappa_v$$

with arbitrary $F_{1*}(B)$ and $F_{2*}(B)$ responsible for the

HS and stringy/tensorial features, respectively

$F_{2*}(B) \neq 0$ for $p \geq 2$.

The framed construction leads to a proper massless spectrum.

Color and Multi-Particle Extensions

W, S and B are allowed to be valued in any associative algebra A.

To make contact with the tensorial boundary theory $A = (Mat_N)^p$ with elements represented by $a^{u_1 \dots u_p}{}_{v_1 \dots v_p}$, $u_i, v_i = 1 \dots N$.

p is the tensor degree of the boundary model

Multi-particle extensions are associated with the semi-simple Coxeter groups. The simplest option with $\mathcal{C} = B_p^{\mathcal{N}}$ is the product of $\mathcal{N}$ of B_p systems

$$B_p^{\mathcal{N}} := \underbrace{B_p \times B_p \times \dots}_{\mathcal{N}} .$$

The limit $\mathcal{N} \to \infty$ along with the graded symmetrization of the product factors expressing the spin-statistics gives the (graded symmetric) multi-particle algebra $M(h(\mathcal{C}))$ of the HS algebra $h(\mathcal{C})$

$M(h(\mathcal{C})) = U(h(\mathcal{C}))$: Hopf algebra.

Klein Operators and Single-Trace Operators

Enlargement of the field spectra of the rank-$p>1$ Coxeter HS models:

$C(Y^n_\alpha;k_v)$ **depend on p copies of oscillators Y^n_α and Klein operators k_v**

Qualitative agreement with enlargement of the boundary operators in tensorial boundary models.

Klein operators of Coxeter reflections permute master field arguments

At $p=2$ the star product of two master fields $C(Y_1,Y_2|x)k_{12}$ gives

$$(C(Y_1,Y_2|x)k_{12})*(C(Y_1,Y_2|x)k_{12})=C(Y_1,Y_2|x)*C(Y_2,Y_1|x)\,.$$

$p=2$ **system: strings of fields with repeatedly permuted arguments**

$$C^n_{string}:=\underbrace{C(Y_1,Y_2|x)*C(Y_2,Y_1|x)*C(Y_1,Y_2|x)\ldots}_{n}\,.$$

are analogous of the single-trace operators in AdS/CFT.

$C(Y_1,Y_2|x)$ **and** $C(Y_1,Y_2|x)*C(Y_2,Y_1|x)$: **single-trace-like**

$C(Y_1,Y_2|x)*C(Y_1,Y_2|x)$: **double-trace-like.**

From Coxeter HS Theory to Strings and Tensor Models

The spectrum of the B_2 HS model is analogous to that of String Theory with the infinite set of Regge trajectories.

B_2- HS theory has parallels with the stringy Gaberdiel-Gopakumar HS models: dependence on $Y^A_{1,2}$ is like having two HS symmetry algebras

B_p−HS models with $p \geq 2$ have two coupling constants.

F_{1*} is analogous to that of the B_1-HS theory.

F_{2*} first appears in the rank-two stringy model and, containing the Klein operators that permute different Y-variables, generates single-trace-like strings of operators and their tensor generalizations.

To establish relation with usual string theory in flat space the limit $F_{2*}/F_{1*} \to \infty$ is most interesting.

Conclusion

Coxeter HS theories

main principle: formal consistency & massless fields in the spectrum

Tensor-like models are natural duals of the rank-p Coxeter HS models

B_2–HS model is conjectured to be string-like

$\mathcal{N} = 4$ SYM is argued to be a natural dual of the B_2–HS model

B_p–Coxeter HS theories have two coupling constants and are formulated in AdS: the stringy B_2-HS models are different from the genuine String Theory in flat space

Multi-particle states of a lower-dimensional model = elementary states in a higher-dimensional (particularly, $10d$ model)

The original $3d$ and $4d$ spinorial theories: branes in the $10d$ theory with the $3d$ HS model as a brick from which the others are composed.

Vector Coxeter HS models in any d can also be introduced

The reason why it is difficult to formulate String Theory in AdS_d is analogous:

a naive attempt to deform the string spectrum to AdS leads to infinite lowest energy since all string modes have to contribute to the momentum generators to ensure that

$$[P_n\,, P_m] \sim \Lambda M_{nm}$$

Idempotent Extension

Let A be an associative algebra with the star product and a set of idempotents

$$\pi_i * \pi_i = \pi_i \,, \qquad \pi_i \in A \,.$$

$$a_i{}^j \in A_i{}^j : \qquad a_i{}^j = \pi_i * a * \pi_j \,, \qquad a \in A \,.$$

The matrix-like composition law in A_π

$$(a * b)_i{}^j = \sum_k a_i{}^k * b_k{}^j$$

A is the algebra of functions of dx, dZ, Z, Y, k_v, x

The set of idempotents π_i has to be $\mathcal{C}$-invariant

The idempotent-extended $\mathcal{C}$-HS equations have the same form with the replacement of $A \to A_{\{\pi\}}$. Idempotent extensions of the Coxeter HS systems describe

lower-dimensional brane-like objects.

Vector-Like and Supersymmetric Models

Fock idempotent in the B_1 $4d$ HS theory

$$\pi_i^{star} = 4 I_i \exp y_{i\alpha} \bar{y}_i^{\alpha}$$

${A_0}^i$**-module describes $3d$ conformal fields= $4d$ singletons:**

Idempotent realization of Klebanov-Polyakov AdS_4/CFT_3

vector model HS holography checked by Giombi and Yin in 2009

B_2 HS model

$4d$ **conformal massless fields are valued in the Fock module π** 2002

$$a_\alpha * \pi = 0\,, \qquad \bar{b}^{\dot\beta} * \pi = 0\,, \qquad \phi_i * \pi = 0\,, \qquad \pi * \bar{a}^{\dot\alpha} = 0\,, \ldots$$

$$[a_\alpha, b^\beta]_* = \delta_\alpha^\beta\,, \qquad [\bar{a}_{\dot\gamma}, \bar{b}^{\dot\beta}]_* = \delta_{\dot\gamma}^{\dot\beta}\,, \qquad \{\phi_i, \bar{\phi}^j\}_* = \delta_i{}^j\,,$$

$i, j = 1, \ldots N$. **Bilinears:** $su(2,2;N)$. **Clifford oscillators: color** $Mat_{2^{2N}}$.

The system is consistent at $N \geq 4$ when $\#_B \leq \#_F$.

$N = 4$ **SYM: the only $N = 4$ massless system with spins $s \leq 1$.**

Higher-Spin Higgsing

To make connection with Strings the most fundamental question is breaking of HS symmetries

Spontaneous breaking of HS symmetries resulting in the massive HS fields is only possible in string-like models with the infinite number of Regge trajectories like B_2 multi-particle theory

The simplest option is to give VEV to a topological field $B(Y_1;Y_2)$

$$B_0 = Y_{iA} \cdot Y_j^A (\alpha k^{ij} + \beta \sigma_1^{ij} + \gamma \delta^{ij})$$

that preserves the AdS symmetry but breaks down the HS one.

Spontaneous symmetry breaking: mixing between the massless rank-one particle module and rank-two current module.

Unitarity

Covariant derivative for a rank-two field $C(Y_1,k_1;Y_2,k_2):=C_{0,1}(Y_1,Y_2)k_2$

$$D_0(C_{0,1}(Y_1,Y_2))=\Big(D^{\mathsf{L}}-h^{\alpha\dot\beta}\Big(y_{1\alpha}\frac{\partial}{\partial\bar y_1^{\dot\beta}}+\frac{\partial}{\partial y_1^{\alpha}}\bar y_{1\dot\beta}-iy_{2\alpha}\bar y_{2\dot\beta}+i\frac{\partial^2}{\partial y_2^{\alpha}\partial\bar y_2^{\dot\beta}}\Big)\Big)C_{0,1}(Y_1,Y_2)$$

$C_{0,1}(Y_1,Y_2)$ is valued in the tensor product of the Y_1-adjoint module and Y_2-twisted adjoint module.

Twisted adjoint module and its tensor products correspond to unitary multi-particle-like states.

Zero-form fields containing an adjoint module a factor do not form a unitary particle-like representation except for the Y_1-independent $C_{1,2}(Y_1,Y_2)$ which decribes I_1 unitary massless states in the Y_2 sector.

Non-singlet states in the adjoint module factors can be truncated away: non-singlet elements of the adjoint factor are never generated.

Vector-Like Models

Fock idempotent in the $4d$ HS theory

$$\pi_i^{star} = 4I_i \exp y_{i\alpha}\bar{y}_i^{\alpha}$$

$$(y_{i\alpha} - i\bar{y}_{i\alpha}) * \pi_i^{star} = 0\,, \qquad \pi_i^{star} * (y_{i\alpha} + i\bar{y}_{i\alpha}) = 0\,.$$

For HS fields carrying matrix indices

$$\pi_i = \pi_i^{star}\pi_i^{color}\,, \qquad \pi_i^{color} = \delta_1^u \delta_v^1\,.$$

${A_0}^i$**-module describes $3d$ conformal fields= $4d$ singletons:**

Idempotent realization of Klebanov-Polyakov AdS_4/CFT_3

vector model HS holography checked by Giombi and Yin in 2009

Idempotent extensions of the Coxeter HS systems describe

lower-dimensional brane-like objects.

$N = 4$ SUSY

$4d$ conformal massless fields are valued in the Fock module π 2002

$$a_\alpha * \pi = 0\,, \qquad \bar{b}^{\dot\beta} * \pi = 0\,, \qquad \phi_i * \pi = 0\,, \qquad \pi * \bar{a}^{\dot\alpha} = 0\,, \ldots$$

$$[a_\alpha, b^\beta]_* = \delta_\alpha^\beta\,, \qquad [\bar{a}_{\dot\gamma}, \bar{b}^{\dot\beta}]_* = \delta_{\dot\gamma}^{\dot\beta}\,, \qquad \{\phi_i, \bar\phi^j\}_* = \delta_i^j\,,$$

$i, j = 1, \ldots N$. Bilinears: $su(2,2;N)$. Clifford oscillators: color $Mat_{2^{2N}}$.

B_2-HS theory contains $y_{i\alpha}$, $\bar{y}_{i\dot\alpha}$. Vacuum π is defined by ϕ_i, $\bar\phi^j$ and

$$a_\alpha = y_{1\alpha} + i y_{2\alpha}\,, \quad b_\alpha = \frac{1}{4i}(y_{1\alpha} - i y_{2\alpha})\,, \quad \bar{a}_{\dot\alpha} = \bar{y}_{1\dot\alpha} - i\bar{y}_{2\dot\alpha}\,, \quad \bar{b}_{\dot\alpha} = \frac{1}{4i}(\bar{y}_{1\dot\alpha} + i\bar{y}_{2\dot\alpha})$$

$4d$ massless conformal fields are valued in the Fock modules.

Reflection $Y_1^A \leftrightarrow Y_2^A$ maps π to the opposite idempotent $\tilde\pi$

$$b_\alpha * \tilde\pi = 0\,, \qquad \bar{a}^{\dot\beta} * \tilde\pi = 0\,, \qquad \bar\phi^i * \tilde\pi = 0\,, \qquad \tilde\pi * \bar{b}^{\dot\alpha} = 0\,, \ldots$$

Both π and $\tilde\pi$ have to be present. Elements $\pi * a * \tilde\pi$ are ill defined: at $N = 0$, $\pi * \tilde\pi = \infty$. Bosons and fermions contribute with opposite signs. The compensation occurs at $N = 4$ when $\#_B = \#_F$. $N = 4$ SYM is the only $N = 4$ massless conformal system with spins $s \leq 1$.