Generating formulation for higher spin fermions

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based on:
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QFTC 2018

Tomsk State Pedagogical Univercity

August 3, 2018

Technique

- $\mathfrak{o}(1, d-1) - \mathfrak{osp}(1|2n)$ Howe duality
- BRST formulation
- Homological reduction

Spinor-tensor fields

Let us introduce Grassmann even variables a_I^a and $\bar{a}_b^J$, where $a, b = 0, ..., d-1$, $I, J = 0, ..., n$ and Grassmann odd variables θ^a satisfying the canonical commutation relations

$$[\bar{a}_a^I, a_J^b] = \delta_J^I \delta_a^b \,, \qquad \{\theta^a, \theta^b\} = 2\eta^{ab} \,,$$

where $\eta^{ab} = (- + \cdots +)$ is the Minkowski tensor. Consider the space $\mathcal{P}_n^d(a_I) = \mathcal{S} \otimes \mathcal{T}$ of spinor-tensor polynomials in a_I^a, where $\mathcal{S}$ is the representation of the Clifford algebra generated by θ^a and $\mathcal{T}$ is the tensor product of tensor algebras generated by a_I^a. Elements of $\mathcal{P}_n^d(a_I)$ have the form

$$\psi(a) = \sum_{m_I} \psi^{\alpha}{}_{a_1 \, \ldots \, a_{m_0} ; \, \ldots\ldots \, ; \, b_1 \, \ldots \, b_{m_{n-1}}} a_0^{a_1} \ldots a_0^{a_{m_0}} \; \ldots \; a_{n-1}^{b_1} \ldots a_{n-1}^{b_{m_{n-1}}} \,,$$

An associative algebra generated by a_I^a, $\bar{a}_b^J$ and θ^a naturally acts on $\mathcal{P}_n^d(a_I)$ as follows

$$a_I^a \psi(a) := a_I^a \psi(a) \,, \qquad \bar{a}_a^I \psi(a) := \frac{\partial}{\partial a_I^a} \psi(a) \,, \qquad \theta^a \psi(a) := \gamma^a \psi(a) \,.$$

Lorentz algebra and orthosymplectic superalgebra

The Lorentz algebra $\mathfrak{so}(1, d-1)$ can be represented on $\mathcal{P}^d_n(a_I)$ by the following basis elements

$$M_{ab} = a_{I\,a}\bar{a}^I_b - a_{I\,b}\bar{a}^I_a + \frac{1}{4}(\theta_a\theta_b - \theta_b\theta_a)\,.$$

It follows that the expansion coefficients in a^a_I transform as Lorentz spinor-tensors.
Simultaneously, the orthosymplectic superalgebra $\mathfrak{osp}(1|2n+2)$ acts on $\mathcal{P}^d_n(a_I)$. The even basis elements are given by

$$T_{IJ} = a^a_I a_{Ja}\,, \quad T_I{}^J = \frac{1}{2}\left(a^a_I \bar{a}^J_a + \bar{a}^J_a a^a_I\right), \quad T^{IJ} = \bar{a}^I_a \bar{a}^{Ja}\,,$$

and odd basis elements are given by

$$\Upsilon_I = a^a_I \theta_a\,, \qquad \Upsilon^I = \bar{a}^I_a \theta^a\,.$$

Poincare algebra

The Poincare algebra $\mathfrak{iso}(d-1,1)$ can be realized on the same set of oscillators. We split the original variables as $a_0^a \equiv x^a$, $a_I^a \equiv a_i^a$, $I > 0$ with $i = 1, ..., n$. Then, translations and Lorentz rotations are given by

$$P_a = \partial_a \,, \qquad M_{ab} = x_a\partial_b - x_b\partial_a + a_{i\,a}\bar{a}^i_b - a_{i\,b}\bar{a}^i_a + \frac{1}{4}(\theta_a\theta_b - \theta_b\theta_a) \,.$$

and naturally act in the space $\mathcal{P}^d_n(x,a)$ of smooth functions in x^a with values in $\mathcal{P}^d_n(a_i)$.

We also introduce special notation for some of $\mathfrak{osp}$ even basis elements

$$\Box \equiv T^{00} = \partial_a\partial^a \,, \qquad D^i \equiv T^{0i} = \bar{a}^a_i\partial_a \,, \qquad D^\dagger_i \equiv T_i{}^0 = a^a_i\partial_a \,,$$

$$N_i^j \equiv T_i{}^j = a^a_i\,\bar{a}_{ja} \qquad i \neq j \,, \qquad N_i \equiv T_i{}^i - \frac{d}{2} = a^a_i\,\bar{a}_{ia} \,,$$

and for odd basis elements

$$\hat{D} \equiv \Upsilon^0 = \theta^a\partial_a \,.$$

One-parameter constraint system

Dirac constraint:

$$\hat{D}\psi = 0\,,$$

gamma-trace condition:

$$\left(\Upsilon^i + \nu^i \Gamma\right)\psi = 0\,, \quad \nu^i = \nu\delta^{1i}\,, \quad i = 1, \ldots, n\,,$$

spin weight and Young symmetry conditions:

$$N_m\psi = s_m\psi\,, \qquad {N_m}^k\psi = 0\,, \quad m, k = 2, \ldots, n\,.$$

Gauge transformation:

$$\delta\psi = \left(D_i^\dagger + \mu_i\right)\chi^i\,, \quad \mu_i = \mu\delta_{i1}\,, \quad i = 1, \ldots, n\,.$$

Here, $\mu, \nu \in \mathbb{R}$, spin weights $s_m \in \mathbb{N}$, and Γ is the extra Clifford element satisfying $\{\Gamma, \theta^a\} = 0$ and $\Gamma^2 = 1$.

Casimir elements

The quadratic and quartic Casimir operator of the Poincare algebra $\mathfrak{iso}(1, d-1)$ evaluated on the constraints above are diagonalized

$$C_2\psi = P^a P_a \psi = 0 \ ,$$

$$C_4\psi = M_{ab}P^b M^{ac} P_c - \frac{1}{2} M^2 P^2 = -\mu^2\nu^2\psi \ .$$

The continuous spin parameter is $\mu\nu$.

Triplet formulation

All differential constraints and gauge transformation can be realized via BRST operator

$$\Omega = \alpha\hat{D} + c_0\Box + c_i D^i + (D_i^\dagger + \mu_i)\frac{\partial}{\partial b_i} - \alpha\alpha\frac{\partial}{\partial c_0} - c_i\frac{\partial}{\partial b_i}\frac{\partial}{\partial c_0}$$

which acts on a subspace singled out by ghost extended algebraic constraints

$$\widetilde{N}_i\Psi = s_i\Psi\ , \qquad \widetilde{N}_i{}^j\Psi = 0 \quad (i<j)\ , \qquad \left(\widetilde{\Upsilon}^i + \nu^i\Gamma\right)\Psi = 0\ ,$$

where

$$\widetilde{N}_i{}^j = N_i{}^j + b_i\frac{\partial}{\partial b_j} + c_i\frac{\partial}{\partial c_j}\ , \quad \widetilde{N}_i = N_i\ , \quad \widetilde{\Upsilon}^i = \Upsilon^i - 2\alpha\frac{\partial}{\partial c_i} + \frac{\partial}{\partial\alpha}\frac{\partial}{\partial b_i}\ .$$

Here we introduced anticommuting ghost variables c_0, c_i, b_i with $i = 1, \dots, n-1$ and commuting ghost variable α with ghosts numbers $\mathrm{gh}(c_0) = \mathrm{gh}(c_i) = \mathrm{gh}(\alpha) = 1$, $\mathrm{gh}(b_i) = -1$.

Lagrangian formulation for helicity spin case

If $\mu = \nu = 0$ the BRST operator Ω is symmetric with respect to the inner product

$$\langle \psi, \varphi \rangle = \int \mathrm{d}^d x \int \mathrm{d}c_0 \langle \psi, \varphi \rangle',$$

where $\langle \cdot, \cdot \rangle'$ is the inner product on the Fock module generated by a_i^a, c_i, b_i from the "vacuum space" $|V\rangle$ which is the irreducible representation of the Clifford algebra generated by θ^a with positive semidefinite inner product making θ^a hermitean (e.g. if we have the representation where θ^a act as (anti)hermitean gamma matrices it would be $\bar{\psi}\gamma^0\varphi$), such that $\frac{\partial}{\partial a_i^a}|V\rangle = \frac{\partial}{\partial c_i}|V\rangle = \frac{\partial}{\partial b_i}|V\rangle$.
So in that case the equations of motion $\Omega\Psi^{(0)} = 0$ follow from the action

$$S = \frac{1}{2}\langle \Psi^{(0)}, \Omega\Psi^{(0)} \rangle.$$

Reductions: metric-like formulation

The metric-like formulation that generalizes the Fang-Fronsdal description of higher spin massless fields can be obtained from the triplet formulation by introducing the additional grading as a homogeneity degree in c_0. Then the triplet BRST operator Ω decomposes as $\Omega = \Omega_{-1} + \Omega_0 + \Omega_1$ with

$$\Omega_{-1} = -\left(\alpha\alpha + c_i \frac{\partial}{\partial b_i}\right)\frac{\partial}{\partial c_0}\,, \quad \Omega_0 = \alpha\hat{D} + c_i D^i + (D_i^\dagger + \mu_i)\frac{\partial}{\partial b_i}\,, \quad \Omega_1 = c_0\Box$$

and we can reduce the theory to the cohomology $H(\Omega_{-1})$ that contains required spinor-tensor fields of the metric-like formulation.

Reductions: metric-like formulation

EOM (for $\mu = \nu = 0$ it's the Fang-Fronsdal-Labastida kinetic operator for mixed-symmetry fermionic fields)

$$\left[\hat{D} - (D_i^\dagger + \mu_i)(\Upsilon^i + \nu^i \Gamma)\right] \psi = 0 \,, \tag{1}$$

are invariant with respect to the gauge transformations

$$\delta\psi = (D_i^\dagger + \mu_i)\chi^i \,,$$

where both fields and (differentially unconstrained) gauge parameters are subjected to the modified trace conditions

$$\boldsymbol{\Upsilon}^{(i}\boldsymbol{\Upsilon}^{j}\boldsymbol{\Upsilon}^{k)}\psi = 0, \qquad \boldsymbol{\Upsilon}^{(i}\chi^{j)} = 0,$$

where $\boldsymbol{\Upsilon}^i = \Upsilon^i + \nu^i \Gamma$.

It is worth noting that the equation (1) can be squared to yield

$$\left[\Box - (D_i^\dagger + \mu_i)D^i + \frac{1}{2}(D_i^\dagger + \mu_i)(D_j^\dagger + \mu_j)(T^{ij} + \nu^i \nu^j)\right]\psi = 0$$

which is the bosonic continuous spin metric-like equation [K. Alkalaev, M. Grigoriev 2017].

Reductions: light-cone formulation

We realize the representation of Clifford algebra with generators $\theta^+, \theta^-, \theta^m\,, m = 1, \ldots, d-2$ as polynomials in θ^+ with coefficients in $\mathfrak{o}(d-2)$ spinors $\psi^{\hat\alpha}$, where Dirac spinor index $\hat\alpha = 1, ..., 2^{d/2-1}$. In other words, "the light-cone spinor is half of the original spinor".
To do the light-cone reduction of the fermionic triplet formulation we introduce the grading

$$\deg a_i^\pm = \pm 2\,, \quad \deg c_i = 1\,, \quad \deg b_i = -1\,, \quad \deg \theta^+ = 2\,, \quad \deg \alpha = 1\,,$$
$$\deg a^m = 0, \qquad \deg c_0 = 0\,.$$

The operator Ω decomposes into the homogeneous degree components as $\Omega = \Omega_{-1} + \Omega_0 + \Omega_1 + \Omega_2 + \Omega_3$, where

$$\Omega_{-1} = p^+ \left(2\alpha \frac{\partial}{\partial\theta^+} + c_i \frac{\partial}{\partial a_i^+} + a_i^- \frac{\partial}{\partial b_i}\right), \quad \Omega_0 = c_0 \Box\,, \quad \ldots\,.$$

Reductions: light-cone formulation

The reduced BRST operator is just

$$\widetilde{\Omega} = c_0 \Box$$

acting on the light-cone algebraic constraints

$$\begin{aligned}
&\Big(\theta^m \frac{\partial}{\partial a_i^m} + \nu^i \Gamma\Big)\psi = 0, \qquad i = 1, \ldots, n \,, \\
&a_i^m \frac{\partial}{\partial a_j^m}\psi = 0, \qquad 2 \leq i < j \leq n \,, \\
&a_i^m \frac{\partial}{\partial a_i^m}\psi = s_i \psi, \qquad i = 2, \ldots, n, \quad \text{(no sum over } i\text{)} \,.
\end{aligned} \tag{2}$$

Thus, the light-cone fields ψ are $\mathfrak{o}(d-2)$ spinor-tensors subjected to the light-cone condition $p^2 = 0$ and the algebraic conditions (2).

Casimir operators of $\mathfrak{iso}(d-2)$

Basis for a stability subalgebra:

$$H^m = \mu \frac{\partial}{\partial a_{1m}}\,, \qquad S^{mn} = a_i^m \frac{\partial}{\partial a_{in}} + \frac{1}{4}\theta^m\theta^n - (m \leftrightarrow n)\,.$$

Eigenvalues of the first two Casimir operators of the $\mathfrak{iso}(d-2)$ are

$$c_2 = -\mu^2\nu^2\,,$$

$$c_4 = -\mu^2\nu^2 \left(\sum_{i=2}^{n} s_i(s_i + d - 1 - 2i) + \left\{ \sum_{i=2}^{n} s_i + \frac{(d-3)(d-4)}{8} \right\} \right)\,.$$

E.g. for $\mu\nu = m, d-2 = 4, n = 2, s_2 = s$ we get, for the bosonic part of the expression, the well known value $-m^2 s(s+1)$ of the square of the Pauli-Lubanski pseudovector for a field of mass m and spin s in 4d space.

Fin

Thanks for attention!